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The Cosmic Microwave Background

If Heinrich Olbers had lived in intergalactic space and had eyes that operated at millimeter wavelengths (admittedly a very large "if"), he would not have formulated Olbers' paradox. At wavelengths of a few millimeters, thousands of times longer than human eyes can detect, most of the light in the universe comes not from the hot balls of gas we call stars, but from the cosmic microwave background (CMB). Unknown to Olbers, the night sky actually *is* uniformly bright – it's just (2.33) ⇒ uniformly bright at a temperature $T_0 = 2.7255\,\mathrm{K}$ rather than at a temperature $T \sim T_\odot \sim 6000\,\mathrm{K}$. The current energy density of the cosmic microwave background,

(2.28) (2.34) ⇒

$$\varepsilon_{\gamma,0} = \alpha T_0^4 = 0.2606\,\mathrm{MeV\,m^{-3}}, \tag{8.1}$$

is only one part in 19 000 of the current critical density. However, since the energy per CMB photon is small ($hf_{\rm mean} = 6.34 \times 10^{-4}\,\mathrm{eV}$), the number density of CMB photons in the universe is large:

(2.35) ⇒

$$n_{\gamma,0} = 4.107 \times 10^8\,\mathrm{m^{-3}}. \tag{8.2}$$

(4.28) + (4.31)

It is particularly enlightening to compare the energy density and number density of photons to those of baryons (that is, protons and neutrons). In the Benchmark Model, the current energy density of baryons is

$\varepsilon_{c,0} = \frac{3c^2H_0^2}{8\pi G} = 4870\,\frac{\mathrm{MeV}}{\mathrm{m}^3}$ ⇒

$$\varepsilon_{\rm bary,0} = \underset{(7.6)}{\Omega_{\rm bary,0}}\,\underset{(4.31)}{\varepsilon_{c,0}} \approx 234\,\mathrm{MeV\,m^{-3}} \simeq 0.048 \times 4870 \tag{8.3}$$

The energy density in baryons today is thus 900 times the energy density in CMB photons. Note, though, that the rest energy of a proton or neutron, $E_{\rm bary} \approx$ 1GeV ≃ 939 MeV, is more than a trillion times the mean energy of a CMB photon. The number density of baryons, therefore, is much lower than the number density of photons:

$$n_{\rm bary,0} = \frac{\varepsilon_{\rm bary,0}}{E_{\rm bary}} \approx \frac{234\,\mathrm{MeV\,m^{-3}}}{939\,\mathrm{MeV}} \approx 0.25\,\mathrm{m^{-3}} \simeq \frac{1\,\mathrm{H}}{4\,\mathrm{m}^3} \tag{8.4}$$

The ratio of baryons to photons in the universe (a number usually designated by the Greek letter η) is, from Equations 8.2 and 8.4,

$$\eta_0 = \frac{n_{\text{bary},0}}{n_{\gamma,0}} \approx \frac{0.25\,\text{m}^{-3}}{4.107 \times 10^{8}\,\text{m}^{-3}} \approx 6.1 \times 10^{-10}. \tag{8.5}$$

Baryons are badly outnumbered by photons in the universe as a whole, by a ratio of 1.6 billion to one. We use instead: $\eta_0^{-1} = \frac{n_{\gamma,0}}{n_{b,0}} \simeq 1.64 \times 10^{9}$ = Photon-to-Baryon ratio today!

8.1 Observing the CMB

Although CMB photons are as common as dirt,[1] Arno Penzias and Robert Wilson were surprised when they serendipitously discovered the cosmic microwave background. At the time of their discovery, Penzias and Wilson were radio astronomers working at Bell Laboratories. The horn-reflector radio antenna they used had previously been utilized to receive microwave signals, of wavelength $\lambda = 7.35$ cm, reflected from an orbiting communications satellite. Turning from telecommunications to astronomy, Penzias and Wilson found a slightly stronger signal than they expected when they turned the antenna toward the sky. They did everything they could think of to reduce "noise" in their system. They even shooed away a pair of pigeons that had roosted in the antenna and cleaned up what they later called "the usual white dielectric" generated by pigeons. (Guano!)

The excess signal remained. It was isotropic and constant with time, so it couldn't be associated with an isolated celestial source. Wilson and Penzias were puzzled until they were put in touch with Robert Dicke and his research group at Princeton University. Dicke had deduced that the universe, if it started in a hot dense state, should now be filled with microwave radiation.[2] In fact, Dicke and his group were in the process of building a microwave antenna when Penzias and Wilson told them that they had already detected the predicted microwave radiation. Penzias and Wilson wrote a paper for *The Astrophysical Journal* in which they wrote, "Measurements of the effective zenith noise temperature of the 20-foot horn-reflector antenna ... at 4080 Mc/s have yielded a value about 3.5 K higher than expected. This excess temperature is, within the limits of our observations, isotropic, unpolarized, and free from seasonal variations (July, 1964–April, 1965). A possible explanation for the observed excess noise temperature is the one given by Dicke, Peebles, Roll, and Wilkinson in a companion letter in this issue." The companion paper by Dicke and his collaborators points out that the radiation could be a relic of an early, hot, dense, and opaque state of the universe.

1 Actually, much commoner than dirt, when you stop to think of it, since dirt is made of baryons.

2 To give credit where it's due, as early as 1948, Ralph Alpher and Robert Herman had predicted the existence of cosmic background radiation with a temperature of "about 5 K." However, their prediction had fallen into obscurity.

$\langle E_{CMB} = h\nu \rangle \simeq 6.34 \times 10^{-4}$ eV

Measuring the spectrum of the CMB, and confirming that it is indeed a blackbody, is not a simple task, even with modern technology. The mean energy per CMB photon (6.34×10^{-4} eV) is tiny compared to the energy required to break up an atomic nucleus ($\sim$ 2 MeV) or even the energy required to ionize an atom ($\sim$ 10 eV). However, the mean photon energy is comparable to the rotational energy of a small molecule such as H_2O. Thus, CMB photons can zip along for more than 13 billion years through tenuous intergalactic gas, then be absorbed a microsecond away from the Earth's surface by a water molecule in the atmosphere. Microwaves with wavelengths shorter than $\lambda \sim 3$ cm are strongly absorbed by water molecules. Penzias and Wilson observed the CMB at a wavelength $\lambda = 7.35$ cm, corresponding to a photon energy $E = 1.7 \times 10^{-5}$ eV, because that was the wavelength of the signals that Bell Labs had been bouncing off orbiting satellites. Thus, Penzias and Wilson were detecting CMB photons far on the low-energy tail of the blackbody spectrum (Figure 2.7), with an energy just 0.027 times the mean photon energy.

The CMB can be measured at wavelengths shorter than 3 cm by observing from high-altitude balloons or from the South Pole, where the combination of cold temperatures and high altitude[3] keeps the atmospheric humidity low. The best way to measure the spectrum of the CMB, however, is to go completely above the damp atmosphere of the Earth. The CMB spectrum was measured accurately over a wide range of wavelengths by the *Cosmic Background Explorer (COBE)* satellite, launched in 1989, into an orbit 900 km above the Earth's surface. The CMB was then mapped at greater angular resolution by the *Wilkinson Microwave Anisotropy Probe (WMAP)*, launched in 2001, and by the *Planck* satellite, launched in 2009. Both *WMAP* and *Planck* were in orbits librating about the L_2 point of the Sun–Earth system, 1.5 million km from the Earth. Multiple results have come from observations of the cosmic microwave background.

1

$T_{CMB,0} = 2.7255$ K $\pm .00006$ K !

Result number one: At any angular position (θ, ϕ) on the sky, the spectrum of the cosmic microwave background is very close to that of an ideal blackbody, as illustrated in Figure 8.1. How close is very close? *COBE* could have detected fluctuations in the spectrum as small as one part in 10^4. No deviations were found at this level within the wavelength range investigated by *COBE*/FIRAS (+ Stalk calib)

2 Dipole

Result number two: The CMB has the dipole distortion in temperature shown in Figure 8.2. That is, although each point on the sky has a blackbody spectrum, in one half of the sky the spectrum is slightly blueshifted to higher temperatures, and in the other half the spectrum is slightly redshifted to lower temperatures.[4] This dipole distortion is a simple Doppler shift, caused by the net

[3] The South Pole is nearly 3 kilometers above sea level; one of the major challenges facing the Amundsen and Scott expeditions was the arduous climb from the Ross Ice Shelf to the central Antarctic plateau.

[4] The stretched "yin-yang" pattern in Figure 8.2 represents the darker, cooler (yin?) hemisphere of the sky and the hotter, brighter (yang?) hemisphere, distorted by the map projection.

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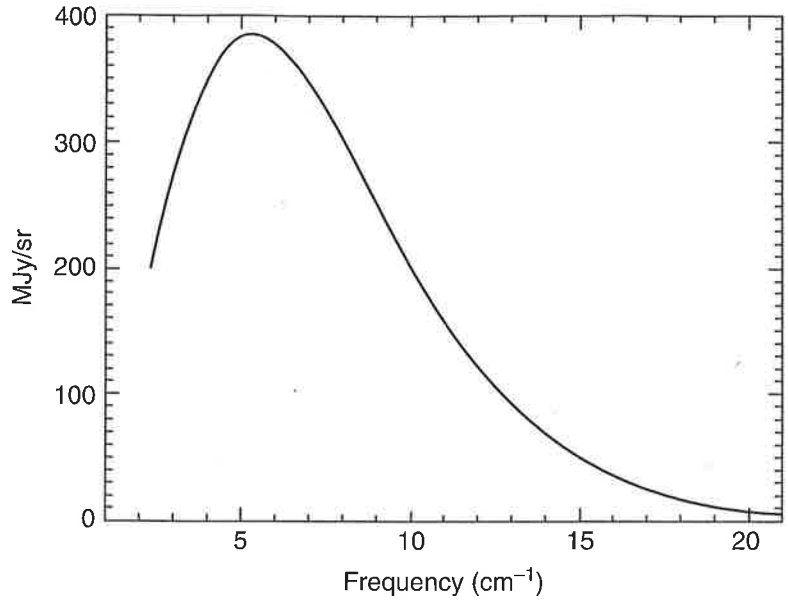


1991 Mather
COBE
$T_0 = 2.7255$
$\pm .0000\varepsilon$
$\frac{\varepsilon}{T_0} \simeq 10^{-5}$

Figure 8.1 The spectrum of the cosmic microwave background, as measured by *COBE*. The uncertainties in the measurement are smaller than the thickness of the line. [Fixsen *et al.* 1996 *ApJ*, **473**, 576]

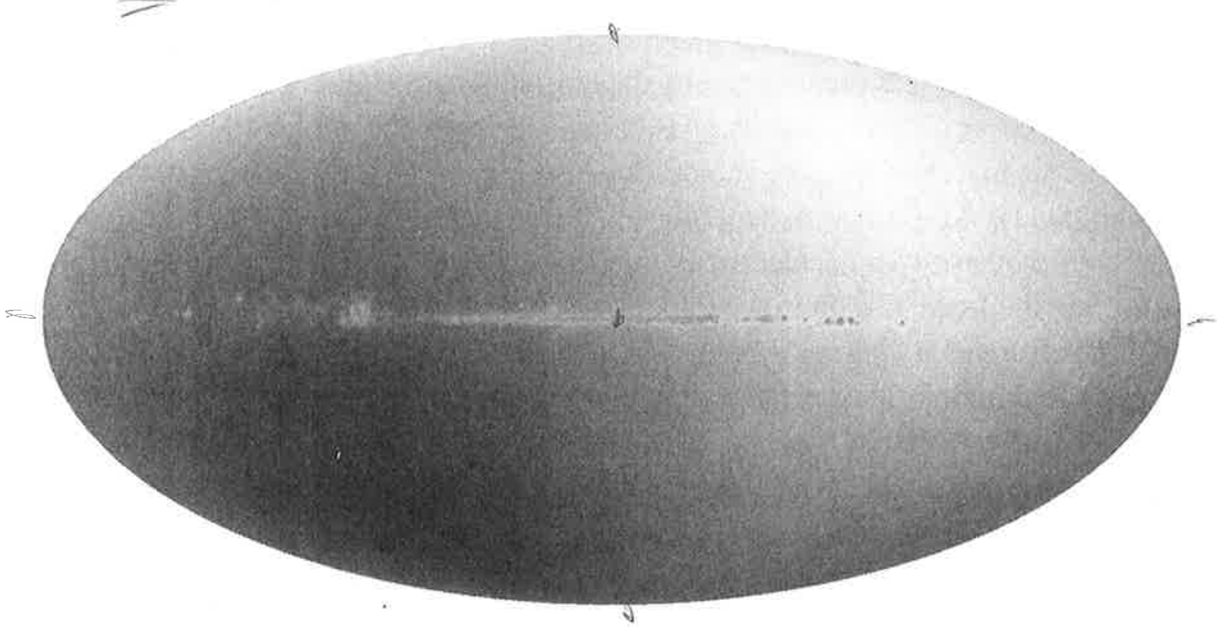
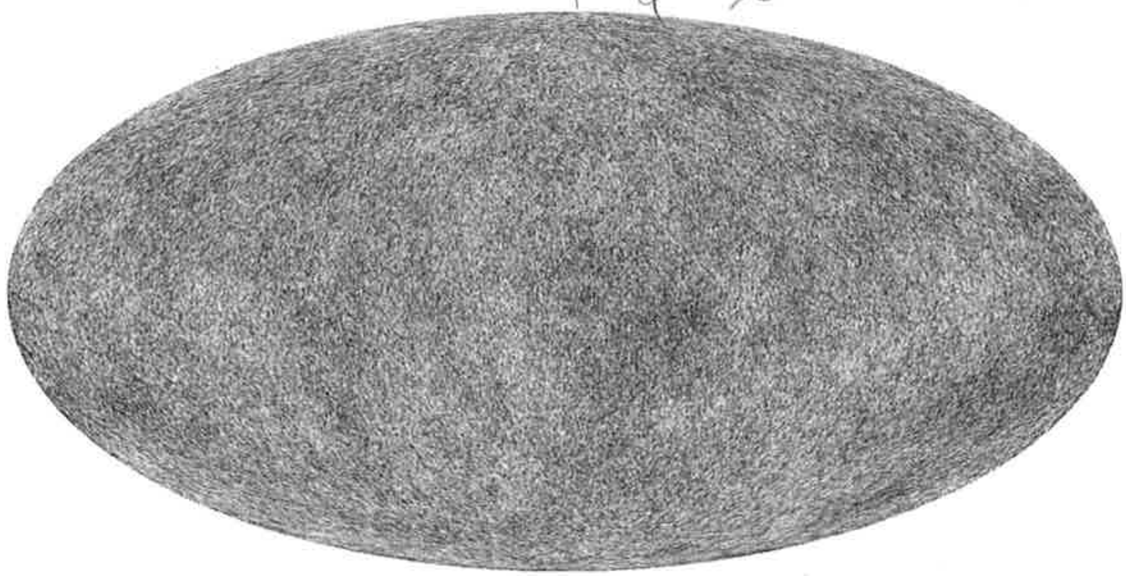


Figure 8.2 The dipole fluctuation in the temperature of the CMB, as measured by *WMAP*. The horizontal band across the middle is non-thermal emission from gas in our own galaxy. [NASA/*WMAP* Science Team]

motion of *WMAP* relative to a frame of reference in which the CMB is isotropic. After correcting for the orbital motion of *WMAP* around the Sun ($v \sim 30\,\mathrm{km\,s^{-1}}$), for the orbital motion of the Sun around the galactic center ($v \sim 235\,\mathrm{km\,s^{-1}}$), and for the orbital motion of our galaxy relative to the center of mass of the Local Group ($v \sim 80\,\mathrm{km\,s^{-1}}$), it is found that the Local Group is moving in the general direction of the constellation Hydra, with a speed $v_{\mathrm{LG}} = 630 \pm 20\,\mathrm{km\,s^{-1}} = 0.0021c$. This peculiar velocity for the Local Group is what you'd expect as the

v_{LG} ($\rightarrow$ Hydra Centaurus Supercluster) $=$ 630 $\pm$20 km s^{-1} $\simeq 0.0021c$
= Great Attractor

$\uparrow \phi$ Dec



$\rightarrow \theta$ RA

Figure 8.3 The fluctuations in temperature remaining in the CMB once the dipole fluctuation and the non-thermal foreground emission from our own galaxy are subtracted. [*Planck*/ESA]

result of gravitational acceleration by the largest lumps of matter in the vicinity of the Local Group. The Local Group is being accelerated toward the Virgo cluster, the nearest big cluster to us. In addition, the Virgo cluster is being accelerated toward the Hydra-Centaurus supercluster, the nearest supercluster to us. The combination of these two accelerations, working over the age of the universe, has launched the Local Group in the direction of Hydra, at 0.2% the speed of light.

(3) *Result number three*: After the dipole distortion of the CMB is subtracted away, the remaining temperature fluctuations, shown in Figure 8.3, are small in amplitude. Let the temperature of the CMB, at a given point on the sky, be $T(\theta, \phi)$. The mean temperature, averaging over all locations, is

$\exists \frac{\Delta T}{T}(\theta,\phi)$

(2.33)

$\simeq \frac{4.0\times 10^{-6}}{2.7255\ldots}$ $\simeq 1.1\times 10^{-5}$

$$\langle T \rangle = \frac{1}{4\pi} \int T(\theta, \phi) \sin\theta \, d\theta \, d\phi = 2.7255\,\mathrm{K}. \tag{8.6}$$

$\simeq \frac{40 \text{ microK}}{2.7255\,\mathrm{K}}$

The dimensionless temperature fluctuation at a given point (θ, ϕ) on the sky is

$$\frac{\delta T}{T}(\theta, \phi) \equiv \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle}. \tag{8.7}$$

After subtraction of the Doppler dipole, the root mean square temperature fluctuation found by *COBE* was

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5}. \tag{8.8}$$

(Given the limited angular resolution of the *COBE* satellite, this excludes the temperature fluctuations on an angular scale $< 10°$.) Even taking into account the blurring from *COBE*'s low resolution, the fact that the CMB temperature varies by only 30 microKelvin across the sky represents a remarkably close approach to isotropy.

The observations that the CMB has a nearly perfect blackbody spectrum and that it is nearly isotropic (once the Doppler dipole is removed) provide strong support for the Hot Big Bang model of the universe. A background of nearly isotropic blackbody radiation is natural if the universe was once hot, dense, opaque, and nearly homogeneous, as it was in the Hot Big Bang scenario. If the universe did not go through such a phase, then any explanation of the cosmic microwave background will have to be much more contrived.

Chapters 3 & 5.4.4 & 9

8.2 Recombination and Decoupling

$z_{EQ}^{rm} = \frac{1}{a_{EQ}} = 3550$

$t_{EQ}^{rm} = 47{,}000\,yr$ (Radiation-Matter Equilibrium)

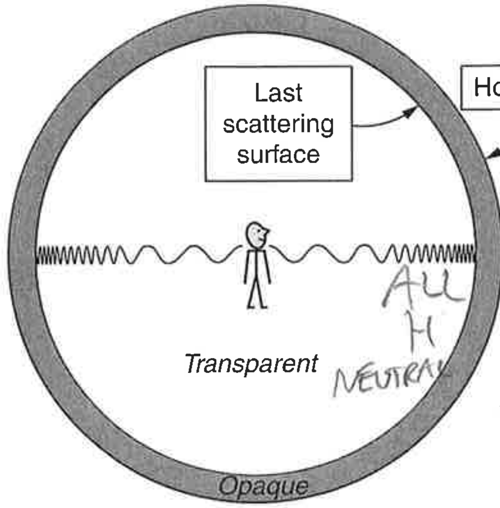
To understand in more detail the origin of the cosmic microwave background, we'll have to examine carefully the process by which the baryonic matter goes from being an ionized plasma to a gas of neutral atoms, and the closely related process by which the universe goes from being opaque to being transparent. To avoid muddle, we will distinguish between three closely related (but not identical) moments in the history of the universe. First, the epoch of *recombination* is the time at which the baryonic component of the universe goes from being ionized to being neutral. Numerically, we define it as the instant in time when the number density of ions is equal to the number density of neutral atoms.[5] Second, the epoch of *photon decoupling* is the time when the rate at which photons scatter from electrons becomes smaller than the Hubble parameter (which tells us the rate at which the universe expands). When photons decouple, they cease to interact with the electrons, and the universe becomes transparent. Third, the epoch of *last scattering* is the time at which a typical CMB photon underwent its last scattering from an electron. Surrounding every observer in the universe is a *last scattering surface*, illustrated in Figure 8.4, from which the CMB photons have been streaming freely, with no further scattering by electrons. The probability that a photon will scatter from an electron is small once the expansion rate of the universe is faster than the scattering rate; thus, the epoch of last scattering is very close to the epoch of photon decoupling.

At $z_{rec} \simeq 1380 \pm 1$, $t = 249{,}000$ yrs

At $z_{dec} \simeq z_{last} \simeq 1090 \pm 1$, $t = 372{,}000$ yrs

To keep things from getting too complicated, we will assume that the baryonic component of the universe consisted entirely of hydrogen at the epoch of recombination. This is not, however, a strictly accurate assumption. Even at the time of recombination, before stars had a chance to pollute the universe with

[5] Cosmologists sometimes grumble that this should really be called the epoch of "combination" rather than the epoch of "recombination," since this is the very first time when electrons and ions combined to form stable atoms.



ONLY $p^+ + e^-$ IONIZED

Chapter 9

$\frac{n_{He}}{n_H+n_{He}} \simeq 0.1$

$Y = \frac{m_{He}}{m_H+m_{He}} \simeq 0.24$

Figure 8.4 An observer is surrounded by a spherical last scattering surface. The photons of the CMB travel straight to us from the last scattering surface, being continuously redshifted.

heavy elements, there was a significant amount of helium present. (In the next chapter, we will examine how this helium was formed in the early universe.) However, the presence of helium is merely a complicating factor. All the significant physics of recombination can be studied in a simplified universe containing no elements other than hydrogen. The hydrogen can take the form of a neutral atom (designated by the letter H), or of a naked hydrogen nucleus, otherwise known as a proton (designated by the letter p). To maintain charge neutrality in this hydrogen-only universe, the number density of free electrons must be equal to that of free protons: $n_e = n_p$. The degree to which the baryonic content of the universe is ionized can be expressed as the fractional ionization X, defined as

(Baryonic) ionization fraction

$$X \equiv \frac{n_p}{n_p + n_{\rm H}} = \frac{n_p}{n_{\rm bary}} = \frac{n_e}{n_{\rm bary}}. \quad \text{ignoring He!} \tag{8.9}$$

The value of X ranges from $X = 1$, when the baryonic content is fully ionized, to $X = 0$, when it consists entirely of neutral atoms.

One useful consequence of assuming that hydrogen is the only element is that there is now a single relevant energy scale in the problem: the ionization energy of hydrogen, $Q = 13.6\,\text{eV}$. A photon with an energy $hf > Q$ is capable of photoionizing a hydrogen atom:

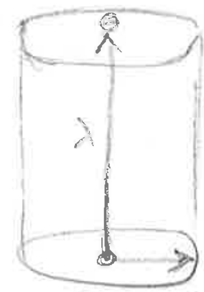
$$\text{H} + \gamma \to p + e^-. \tag{8.10}$$

This reaction can run in the opposite direction, as well; a proton and an electron can undergo *radiative recombination*, forming a bound hydrogen atom while a photon carries away the excess energy:

$$p + e^- \to \text{H} + \gamma. \tag{8.11}$$

In a universe containing protons, electrons, and photons, the fractional ionization X will depend on the balance between photoionization and radiative recombination.

Let's travel back in time to a period before the epoch of recombination. For concreteness, let's choose the moment when $a = 10^{-5}$, corresponding to a redshift $z = 10^5$. (In the Benchmark Model, this scale factor was reached when the universe was seventy years old.) The temperature of the background radiation at this time was $T \approx 3 \times 10^5\,\mathrm{K}$, and the average photon energy was $hf_{\mathrm{mean}} \approx 2.7kT \approx 60\,\mathrm{eV}$, in the extreme ultraviolet. With such a high energy per photon, and with 1.6 billion photons for every baryon, any hydrogen atoms that happened to form by radiative recombination were very short-lived; almost immediately, they were blasted apart into their component electron and proton by a high-energy photon. At early times, then, the fractional ionization of the universe was very close to $X = 1$. $\eta^{-1} \simeq 1.6 \times 10^9$ (= PHOTON / BARYON)



When the universe was fully ionized, photons interacted primarily with electrons, and the main interaction mechanism was Thomson scattering:

$$\gamma + e^- \rightarrow \gamma + e^-. \tag{8.12}$$

The scattering interaction is accompanied by a transfer of energy and momentum between the photon and electron. The cross-section for Thomson scattering is $\pi r_e^2 \equiv \sigma_e = 6.65 \times 10^{-29}\,\mathrm{m}^2$. The mean free path of a photon – that is, the mean distance it travels before scattering from an electron – is $\sigma_e = \pi r_e^2$

$$\text{m.f.p} = \text{mean free path} = \underset{\text{def!}}{\lambda \equiv \frac{1}{n_e \sigma_e}}. \Longleftrightarrow n_e = \frac{1}{\lambda \sigma_e} = \frac{1}{\lambda \pi r_e^2} = \frac{1}{\text{Vol of Cylinder that } e^- \text{ travels per sec}} \tag{8.13}$$

Since photons travel with a speed c, the rate at which a photon undergoes scattering interactions is

$$\Gamma = \frac{c}{\lambda} = n_e \sigma_e c. \quad [\text{time}^{-1}] \tag{8.14}$$

$r_e \simeq 4.6 \times 10^{-15}\,\mathrm{m}$

When the baryonic component of the universe is fully ionized, $n_e = n_p = n_{\mathrm{bary}}$. Currently, the number density of baryons is $n_{\mathrm{bary},0} = 0.25\,\mathrm{m}^{-3}$. ← (8.4) The number density of conserved particles, such as baryons, goes as $1/a^3$, so when the early universe was fully ionized, the free electron density was

$$n_e = n_{\mathrm{bary}} = \frac{n_{\mathrm{bary},0}}{a^3} = n_{B,0} \cdot (1+z)^3 \tag{8.15}$$

and the scattering rate for photons was

$$\Gamma = \frac{n_{\mathrm{bary},0}\sigma_e c}{a^3} = \frac{5.0 \times 10^{-21}\,\mathrm{s}^{-1}}{a^3}. \tag{8.16}$$

This means, for instance, that at $a = 10^{-5}$, photons scattered from electrons at a rate $\Gamma = 5.0 \times 10^{-6}\,\mathrm{s}^{-1}$, about three times a week. (1 week $= 7 \times 24 \times 3600 \simeq 6.05 \times 10^5\,\mathrm{s}$)

At first: $\Gamma > H \Rightarrow$ e^-, p^+ are thermally coupled

$R = \frac{c}{H}$

The photons remain coupled to the electrons as long as their scattering rate, Γ, is larger than H, the rate at which the universe expands; this is equivalent to saying that their mean free path λ is shorter than the Hubble distance c/H. As long as photons scatter frequently from electrons, the photons and electrons remain at the same temperature T (and thanks to the electrons' interactions with protons, the protons also have the same temperature). When the photon scattering rate Γ drops below H, then the electrons are being diluted by expansion more rapidly than the photons can interact with them. The photons then decouple from the electrons and the universe becomes transparent. Once the photons are decoupled from the electrons and protons, the baryonic portion of the universe is no longer compelled to have the same temperature as the cosmic microwave background. During the early stages of the universe ($a < a_{rm} \approx 2.9 \times 10^{-4}$) the universe was radiation dominated, and the Friedmann equation was

When $\Gamma < H$ e^- & CMB-$h\nu$ decouple (i.e. no longer in T-equilibrium)

This is when the $a_m \propto (1+z)^{-3}$, $a_r \propto (1+z)^{-4}$, $T_r \propto (1+z)^{+1}$, $T_m \propto (1+z)^{>+1}$ behavior becomes first visible

Fr. EQ (5.81) with Radiation only

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4}. \tag{8.17}$$

Thus, the Hubble parameter was

$$H = \frac{H_0 \Omega_{r,0}^{1/2}}{a^2} = \frac{2.1 \times 10^{-20}\,\mathrm{s}^{-1}}{a^2}. \tag{8.18}$$

$H^{-1}(z \simeq 10^5) \simeq 4.8 \times 10^9\,\mathrm{sec} \simeq / 31.56 \times 10^6\,\mathrm{s/yr} \Rightarrow T = H^{-1} \simeq 150\,\mathrm{yr}$

This means, for instance, that at $a = 10^{-5}$, the Hubble parameter was $H = 2.1 \times 10^{-10}\,\mathrm{s}^{-1}$. Since this is much smaller than the scattering rate $\Gamma = 5.0 \times 10^{-6}\,\mathrm{s}^{-1}$ at the same scale factor, the photons were well coupled to the electrons and protons.

If hydrogen had remained ionized (and note the qualifying *if*), then photons would have remained coupled to the electrons and protons until a relatively recent time. Taking into account the transition from a radiation-dominated to a matter-dominated universe, and the resulting change in the expansion rate, we can compute that *if* hydrogen had remained fully ionized, then decoupling would have taken place at a scale factor $a \approx 0.0254$, corresponding to a redshift $z \approx 38$ $= \frac{1}{0.0254}$ and a CMB temperature $T \approx 110\,\mathrm{K}$. However, at such a low temperature, the CMB photons are too low in energy to keep the hydrogen ionized. Thus, the decoupling of photons is not a gradual process, caused by the continuous lowering of free electron density as the universe expands. Rather, it is a relatively sudden process, caused by the plummeting of free electron density during the epoch of recombination, as electrons combined with protons to form hydrogen atoms.

8.3 The Physics of Recombination

When does recombination, and the consequent photon decoupling, take place? It's easy to do a quick and dirty approximation of the recombination temperature. Recombination, one could argue, must take place when the mean energy per

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photon of the cosmic microwave background falls below the ionization energy of hydrogen, $Q = 13.6\,\text{eV}$. When this happens, the average CMB photon is no longer able to photoionize hydrogen. Since the mean CMB photon energy is $\sim 2.7kT$, this line of argument would indicate a recombination temperature of

$E_{mean} = \frac{\varepsilon_\gamma}{n_\gamma} \Leftarrow \frac{(2.28+2.29)}{(2.31+2.32)} \Downarrow E_{mean} = \frac{\pi^4}{37} kT \simeq 2.7kT$

$$T_{\text{rec}} \sim \frac{Q}{2.7k} \sim \frac{13.6\,\text{eV}}{2.7(8.6 \times 10^{-5}\,\text{eV}\,\text{K}^{-1})} \sim 60\,000\,\text{K}. \tag{8.19}$$

Alas, this crude approximation is a little *too* crude to be useful. It doesn't take into account the fact that CMB photons are not of uniform energy – a blackbody spectrum has an exponential tail (see Figure 2.7) trailing off to high energies. Although the mean photon energy is $2.7kT$, about one photon in 500 will have $E > 10kT$, one in 3 million will have $E > 20kT$, and one in 30 billion will have $E > 30kT$. Although extremely high energy photons make up only a tiny fraction of the CMB photons, the total number of CMB photons is enormous, with 1.6 billion photons for every baryon. The vast swarms of photons that surround every newly formed hydrogen atom greatly increase the probability that the atom will collide with a photon from the high-energy tail of the blackbody spectrum, and be photoionized.

$\frac{n_\gamma}{n_b} \sim 1.6 \times 10^9$ = photon/baryon ratio η^{-1}

Thus, we expect the recombination temperature to depend on the baryon-to-photon ratio η as well as on the ionization energy Q. An exact calculation of the fractional ionization X, as a function of η and T, requires a smattering of statistical mechanics. Let's start with the reaction that determines the value of X in the early universe:

$\boxed{\eta = \frac{n_b}{n_\gamma}}$

$X \equiv \frac{n_p}{n_p + n_H} = f_{ion} =$ % of free p^+

$$\text{H} + \gamma \rightleftharpoons p + e^-. \tag{8.20}$$

Our calculations will be simplified by the fact that at the time of recombination, the photons, electrons, protons, and hydrogen atoms are in a state of *thermal equilibrium*. Saying that several types of particle are in thermal equilibrium with each other is equivalent to saying that they all have the same temperature T. For instance, in the air around you, the N_2 molecules and O_2 molecules are in thermal equilibrium with each other because of their frequent collisions, and have the same temperature, $T \approx 300\,\text{K}$.[6]

$\Leftrightarrow$ T = same $\forall$

Further simplification comes from the fact that each particle type (photon, electron, proton, or hydrogen atom) is in a state of *kinetic* equilibrium. Saying that a particular type of particle is in kinetic equilibrium is equivalent to saying that the distribution of particle momentum p and energy E is given either by a Fermi–Dirac distribution (if the particles are fermions, with half-integral spin) or by a Bose–Einstein distribution (if the particles are bosons, with integral spin). Suppose, for instance, that particles of type x have a mass m_x. Let $n_x(p)dp$ be the number density of x particles with momentum in the range $p \to p + dp$. If the particles are in kinetic equilibrium at temperature T, then

[6] Slightly cooler if you are using this book for recreational reading as you ski across the Antarctic plateau.

Fermi Dirac + or } distribution
Bose Einstein − }
= a momentum distrib.

$$n_x(p)dp = g_x \frac{4\pi}{h^3} \frac{p^2 dp}{\exp([E - \mu_x]/kT) \pm 1}, \tag{8.21}$$

where the $+$ sign is chosen for a Fermi–Dirac distribution, and the $-$ sign is chosen for a Bose–Einstein distribution. In Equation 8.21, the factor g_x is the *statistical weight* of the particle; for instance, photons, electrons, protons, and neutrons all have $g = 2$, corresponding to their two different spin states. The factor μ_x is the *chemical potential* for particles of type x. As with other forms of potential energy, such as the gravitational potential, we are particularly interested in the difference in chemical potential between two possible states. For instance, in the reaction given by Equation 8.20, if $\mu_{\rm H} + \mu_\gamma > \mu_p + \mu_e$, then the reaction runs preferentially from the higher energy state ($\rm H + \gamma$) to the lower energy state ($p + e^-$), and photoionizations outnumber radiative recombinations. Conversely, if $\mu_{\rm H} + \mu_\gamma < \mu_p + \mu_e$, the reaction runs preferentially in the opposite direction, and radiative recombinations outnumber photoionizations.

IONIZATION
$H + \gamma \to p^+ + e^-$
IF
$\mu_H + \mu_\gamma > \mu_p + \mu_e$
RECOMBINATION
$p^+ + e^- \to H + \gamma$
IF:
$\mu_p + \mu_e > \mu_H + \mu_\gamma$

For photons, the relation between energy, momentum, and frequency is very simple: $E = pc = hf$. Photons are bosons with a statistical weight $g_\gamma = 2$ and a chemical potential $\mu_\gamma = 0$. This means that the number density of photons as a function of frequency f is

MOMENTUM OF PHOTON: $p = \frac{h\nu}{c} = \frac{E}{c}$ $\Rightarrow p = \frac{h}{\lambda} \Leftrightarrow p\lambda = h$

$g_x = 2$ — (8.21) with $p = \frac{h\nu}{c} = \frac{hf}{c}$

$$n_\gamma(f)df = \frac{8\pi}{c^3} \frac{f^2 df}{\exp(hf/kT) - 1}, \tag{8.22}$$

PLANCK'S BLACK BODY EQ! ⇔ (2.30)

CLASSICAL MOMENTUM
$p = mv$ & mc } ⇒
$E = mc^2$
$\Leftrightarrow p = E/c$

which we have already encountered as the blackbody formula of Equation 2.30. Integrating Equation 8.22 over all photon frequencies gives the total number density of photons:

(2.31) } + (2.32) } ⇒

$$n_\gamma = \frac{2.4041}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 = 0.2436\left(\frac{kT}{\hbar c}\right)^3. \tag{8.23}$$

At the time of recombination, electrons, protons, and hydrogen atoms all had $mc^2 \gg kT$, and thus had highly nonrelativistic thermal speeds. When particles of type x are highly nonrelativistic, we can safely use the approximation $p \approx m_x v$ and

EINST + NEWT = ENERGY + MOMENTUM

$$E \approx m_x c^2 + \frac{1}{2} m_x v^2 \approx m_x c^2 + \frac{p^2}{2m_x}. \tag{8.24}$$

(NON-REL: EINSTEIN << NEWTON)

Substituting these values into Equation 8.21, we find that for particles with $m_x c^2 - \mu_x \gg kT$,

(8.21) } + (8.24) } ⇒

$$n_x(p)dp = g_x \frac{4\pi}{h^3 c^3} \exp\left(\frac{-m_x c^2 + \mu_x}{kT}\right) \exp\left(-\frac{p^2}{2m_x kT}\right) p^2 dp, \tag{8.25}$$

MAXWELL BOLTZMANN DISTRIBUTION

representing a Maxwell–Boltzmann distribution of particle speeds. Integrated over all particle momenta, the total number density of the nonrelativistic x particles is

$\int_0^\infty (8.25)\,dp \Rightarrow$

$$n_x = g_x \left(\frac{m_x kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{-m_x c^2 + \mu_x}{kT}\right), \tag{8.26}$$

regardless of whether they are bosons or fermions.

In general, the chemical potential μ_x for particles other than photons will be nonzero. However, at the time of recombination, we can make the further simplifying assumption that the reaction

→ Photoionization ($z \gtrsim 1090$) IONIZED

NEUTRAL ($z \lesssim 1090$) Recombination ←

$$\mathrm{H} + \gamma \rightleftharpoons p + e^- \tag{8.27}$$

was in *chemical* equilibrium. Saying that a reaction is in chemical equilibrium is equivalent to saying that the reaction rate going from left to right balances the reaction rate going from right to left.[7] In the case of Equation 8.27, for instance, this means that within a given volume of hydrogen gas there will be one radiative recombination, on average, for every photoionization. When chemical equilibrium holds true, the sum of the chemical potentials must be equal on both sides of the equation. For Equation 8.27, given that $\mu_\gamma = 0$, this means that $\mu_\mathrm{H} = \mu_p + \mu_e$.

Using Equation 8.26 to find n_H, n_p, and n_e, and assuming that $\mu_\mathrm{H} = \mu_p + \mu_e$, we find an equation that relates the number density of hydrogen atoms, free protons, and free electrons, as long as photoionization remains in equilibrium with radiative recombination:

Use (8.26) for x: $\frac{n_H}{n_p n_e}$ or $\frac{(8.26 \text{ for } n_H)}{(8.26 \text{ for } n_p)(8.26 \text{ for } n_e)}$.

$$\frac{n_\mathrm{H}}{n_p n_e} = \frac{g_\mathrm{H}}{g_p g_e} \left(\frac{m_\mathrm{H}}{m_p m_e}\right)^{3/2} \left(\frac{kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{[m_p + m_e - m_\mathrm{H}]c^2}{kT}\right). \tag{8.28}$$

Equation 8.28 can be simplified further. First, since the mass of an electron is small compared to that of a proton, we can set $m_\mathrm{H}/m_p = 1$. Second, the binding energy $Q = 13.6\,\mathrm{eV}$ is given by the formula $\boxed{(m_p + m_e - m_\mathrm{H})c^2 = Q}$. The statistical weights of the proton and electron are $g_p = g_e = 2$, while the statistical weight of a hydrogen atom is $g_\mathrm{H} = 4$. Thus, the factor $g_\mathrm{H}/(g_p g_e)$ can be set equal to one. The resulting equation,

Einstein $E = mc^2 \Rightarrow Q = \Delta E = \Delta m c^2$

$$\frac{n_\mathrm{H}}{n_p n_e} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kT}\right), \tag{8.29}$$

is called the *Saha equation*, after the astrophysicist Meghnad Saha, who derived it while studying ionization in stellar atmospheres.

Our next job is to convert the Saha equation into a relation between X, T, and η. From the definition of X (Equation 8.9), we can make the substitution

(8.9): $X \equiv \frac{n_p}{n_p + n_H} = \frac{1}{1 + n_H/n_p} \Downarrow$

$$n_\mathrm{H} = \frac{1-X}{X} n_p \simeq \frac{1-X}{X} n_e \tag{8.30}$$

$1 + n_H/n_p = \frac{1}{X} \Rightarrow \frac{n_H}{n_p} = \frac{1}{X} - 1$ ↑ (since universe = charge-neutral!)

[7] Although this type of equilibrium is conventionally called "chemical" equilibrium, it can apply to nuclear reactions (as we'll see in the next chapter) as well as to chemical reactions.

and from the requirement of charge neutrality, we can make the substitution $n_e = n_p$. This yields

ne

(8.29) (8.30) $\Longrightarrow$

$$\frac{1-X}{X} = n_p \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kT}\right). \tag{8.31}$$

To eliminate n_p from the above equation, we recall that $\eta \equiv n_{\rm bary}/n_\gamma$. In a universe where hydrogen is the only element, and a fraction X of the hydrogen is in the form of naked protons, we may write $n_p \equiv X . n_e \Leftarrow$ (8.9) with $n_e \equiv n_p$

η = Inverse (Photon/Baryon ratio)

$$\eta = \frac{n_p}{Xn_\gamma} = \frac{n_e}{n_\gamma} = \left(\frac{\text{Photon}}{\text{Baryon}}\right)^{-1} \tag{8.32}$$

Since the photons have a blackbody spectrum, with a photon number density n_γ given by Equation 8.23, we can combine Equations 8.32 and 8.23 to find

(8.32) $n_p = X\eta n_\gamma$
(8.23) $n_\gamma = \ldots$ $\Longrightarrow$

$$n_p = 0.2436X\eta \left(\frac{kT}{\hbar c}\right)^3. \tag{8.33}$$

Substituting Equation 8.33 back into Equation 8.31, we finally find the desired equation for X in terms of T and η:

(8.33) + (8.31) $\Longrightarrow$

$$\frac{1-X}{X^2} = 3.84\eta \left(\frac{kT}{m_e c^2}\right)^{3/2} \exp\left(\frac{Q}{kT}\right). \tag{8.34}$$

$SX^2 + X - 1 \Longleftrightarrow (ax^2 + bx + c)$

This is a quadratic equation in X, whose positive root is

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$X = \frac{-1 + \sqrt{1+4S}}{2S}, \tag{8.35}$$

where

$$S(T,\eta) = 3.84\eta \left(\frac{kT}{m_e c^2}\right)^{3/2} \exp\left(\frac{Q}{kT}\right). \tag{8.36}$$

$\eta^{-1} = 1.6 \times 10^9$

If we define the moment of recombination as the instant when $X = \frac{1}{2}$, then (assuming $\eta = 6.1 \times 10^{-10}$) the recombination temperature is

$$\boxed{kT_{\rm rec} = 0.324\,{\rm eV} = \frac{Q}{42}}. \tag{8.37}$$

See Fig 8.5

Because of the exponential dependence of S upon the temperature, the exact value of η doesn't strongly affect the value of $T_{\rm rec}$. On the Kelvin scale, $kT_{\rm rec} = 0.324\,{\rm eV}$ corresponds to a temperature $T_{\rm rec} = 3760\,{\rm K}$, slightly higher than the melting point of tungsten.[8] The temperature of the universe had a value $T = T_{\rm rec} = 3760\,{\rm K}$ at a redshift $z_{\rm rec} = 1380$, when the age of the universe, in the Benchmark Model, was $t_{\rm rec} = 250\,000\,{\rm yr}$. Age

$z_{\rm rec} = 1380$

Recombination was not an instantaneous process; it happened sufficiently gradually that at any given instant, the assumption of kinetic and chemical

$z_{\rm CMB} = 1090 \pm 1$ (Planck 2016)

[8] Not that there was any tungsten around back then to be melted.

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IONIZED

$p^+ + e^-$

NEUTRAL

$H + \gamma$

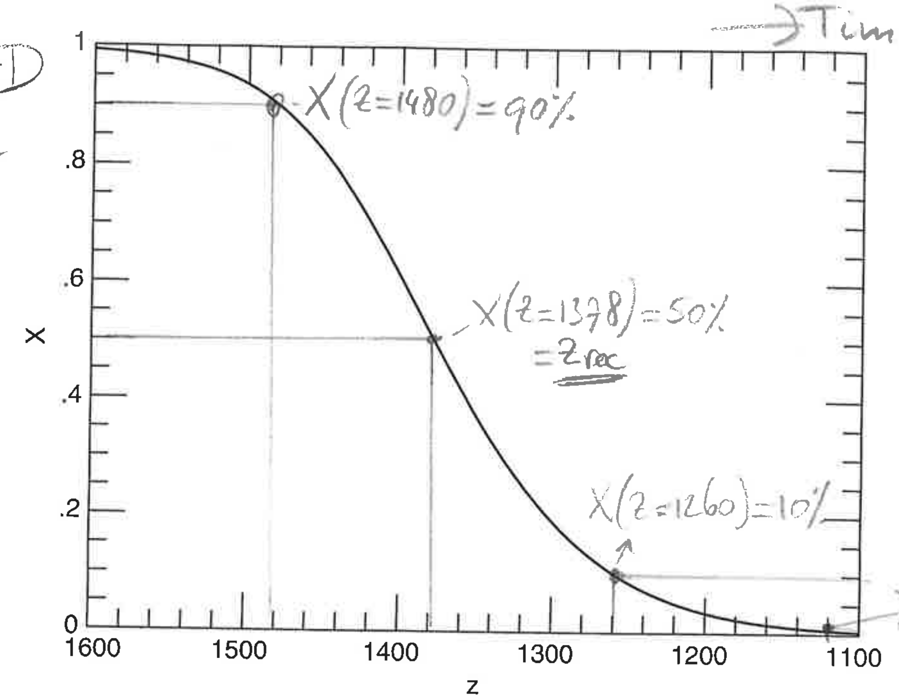

Figure 8.5 Fractional ionization X as a function of redshift during the epoch of recombination. A baryon-to-photon ratio $\eta = 6.1 \times 10^{-10}$ is assumed. Redshift decreases, and thus time increases, from left to right. $\eta^{-1} \simeq 1.6 \times 10^9 = \frac{\text{PHOTON}}{\text{BARYON}}$ ratio.

equilibrium is a reasonable approximation. However, as shown in Figure 8.5, it did proceed fairly rapidly by cosmological standards. The fractional ionization went from $X = 0.9$ at a redshift $z = 1480$ to $X = 0.1$ at a redshift $z = 1260$. In the Benchmark Model, the time that elapses from $X = 0.9$ to $X = 0.1$ is $\Delta t \approx 70\,000\text{ yr} \approx 0.28 t_{\text{rec}}$.

Since the number density of free electrons drops rapidly during the epoch of recombination, the time of photon decoupling comes soon after the time of recombination. The rate of photon scattering, when the hydrogen is partially ionized, is

(8.9) $\frac{n_e}{n_b} = X \Rightarrow n_e = X n_b$

(8.14) + (8.15) $\Rightarrow n_e = n_{b,0}(1+z)^3$

$$\Gamma(z) = n_e(z)\sigma_e c = X(z)(1+z)^3 n_{\text{bary},0}\sigma_e c. \tag{8.38}$$

Using $\Omega_{\text{bary},0} = 0.048$, the numerical value of the scattering rate is

$\Omega_{\text{bary},0} = \frac{\varepsilon_{b,0}}{\varepsilon_{c,0}}$

$$\Gamma(z) = 5.0 \times 10^{-21}\,\text{s}^{-1} X(z)(1+z)^3. \tag{8.39}$$

BUT REALLY USING $n_b = 0.25\,\text{m}^{-3}$ (8.4) I

While recombination is taking place, the universe is matter-dominated, so the Hubble parameter is given by the relation

Fr. Eq (5.81) with Matter dominated $\Rightarrow$

using $H_0 = 68$ km/s/Mpc^{-1} $= 2.20 \times 10^{-18}$ sec since 1 Mpc $= 206265 \times 10^6 \times 149.6 \times 10^6$ km (AU)

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} = \Omega_{m,0}(1+z)^3. \tag{8.40}$$

Using $\Omega_{m,0} = 0.31$, the numerical value of the Hubble parameter during the epoch of recombination is

$\sqrt{\Omega_{m,0}} = 0.566$

$$H(z) = 1.23 \times 10^{-18}\,\text{s}^{-1}(1+z)^{3/2}. \tag{8.41}$$

II

$\Omega_{m,0} \simeq 0.32$ (Planck 2016)

The redshift of photon decoupling is found by setting $\Gamma = H$, or (combining Equations 8.39 and 8.41),

see Fig 8.5 ↓

$$1 + z_{\text{dec}} = \frac{39.3}{X(z_{\text{dec}})^{2/3}}. \Longrightarrow z_{dec} \simeq 1120 \tag{8.42}$$

where $X(z_{dec}) \simeq 0.0066$

Using the value of $X(z)$ given by the Saha equation (shown in Figure 8.5), the redshift of photon decoupling is found to be $z_{\text{dec}} = 1120$. In truth, the exact redshift of photon decoupling is somewhat smaller than this value. The Saha equation assumes that the reaction $\text{H} + \gamma \rightleftharpoons p + e^-$ is in equilibrium. However, when Γ starts to drop below H, the photoionization reaction is no longer in equilibrium. As a consequence, at redshifts smaller than ~ 1200, the fractional ionization X is larger than would be predicted by the Saha equation, and the decoupling of photons is therefore delayed. Without going into the details of the nonequilibrium physics, let's content ourselves by quoting the result $z_{\text{dec}} = 1090$, corresponding to a temperature $T_{\text{dec}} = 2970\,\text{K}$, when the age of the universe was $t_{\text{dec}} = 371\,000$ yr in the Benchmark Model.

(Planck 2016) $z_{dec} \simeq 1090 \pm 1.0$! $= z_{last\ scatter}$ ⟹ z_{dec} is "The surface of last scattering"

$T_{dec} = 1090 \times 2.7255 = 2974\,\text{K}$

When we examine the CMB with our microwave antennas, the photons we collect have been traveling straight toward us since the last time they scattered from a free electron. During a brief time interval $t \to t + dt$, the probability that a photon undergoes a scattering is $dP = \Gamma(t)dt$, where $\Gamma(t)$ is the scattering rate at time t. Thus, if we detect a CMB photon at time t_0, the expected number of scatterings it has undergone since an earlier time t is

$$\tau(t) = \int_t^{t_0} \Gamma(t)dt. \tag{8.43}$$

Also Planck 2016 Optical depth to reionization: $\tau_{reion} \simeq 0.058 \pm .013$ ⇓ $z_{reion} \simeq 8.2 \pm 1.3$

The dimensionless number τ is the *optical depth*. The time t for which $\tau = 1$ is the *time of last scattering*, and represents the time that has elapsed since a typical CMB photon last scattered from a free electron. If we change the variable of integration in Equation 8.43 from t to a, we find that

$$\tau(a) = \int_a^1 \Gamma(a)\frac{da}{\dot{a}} = \int_a^1 \frac{\Gamma(a)}{H(a)}\frac{da}{a}, \quad \text{since } H = \frac{\dot{a}}{a} \tag{8.44}$$

using the fact that $H = \dot{a}/a$. Alternatively, we can find the optical depth as a function of redshift by making the substitution $1 + z = 1/a$:

(8.39) $\Gamma(z) = 5\cdot 10^{-21} X(z)(1+z)^3$
(8.41) $H(z) = 1.23 \cdot 10^{-18}(1+z)^{3/2}$ ⟹

$$\tau(z) = \int_0^z \frac{\Gamma(z)}{H(z)}\frac{dz}{1+z} = 0.0041 \int_0^z X(z)(1+z)^{1/2}dz. \tag{8.45}$$

Here, we have made use of Equations 8.39 and 8.41. As it turns out, the last scattering of a typical CMB photon occurs after the photoionization reaction $\text{H} + \gamma \rightleftharpoons p + e^-$ falls out of equilibrium, so the Saha equation doesn't strictly apply. To sufficient accuracy for our purposes, we can state that the redshift of last scattering was comparable to the redshift of photon decoupling: $z_{\text{ls}} \approx z_{\text{dec}} \approx 1090$. Not all the CMB photons underwent their last scattering

← Planck 2016

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Table 8.1 Events in the early universe.

Event	*Redshift*	*Temperature (K)*	*Time (Myr)*	τ
Radiation–matter equality	3440	9390	0.050	$\gg 1$ ($\sim\infty$)
Recombination	1380	3760	0.25	
Photon decoupling	1090	2970	0.37	
Last scattering	1090	2970	0.37	$\equiv 1.00$
Reionization	8 ± 1	24.5	640.	0.058 ± 0.013
Today	0	2.7255	13820.	$\equiv 0.$

simultaneously; the universe doesn't choreograph its microphysics that well. If we scoop up two photons from the CMB, one may have undergone its last scattering at $z = 1140$, while the other may have scattered more recently, at $z = 1040$. Thus, the "last scattering surface" is really more of a "last scattering layer"; just as we can see a little way into a fog bank here on Earth, we can see a little way into the "electron fog" that hides the early universe from our direct view.

The relevant times of various events around the time of recombination are shown in Table 8.1. For purposes of comparison, the table also contains the time of radiation–matter equality, emphasizing the fact that recombination, photon decoupling, and last scattering took place when the universe was matter-dominated. When we look at the cosmic microwave background, we are getting an intriguing glimpse of the universe as it was when it was only one part in 37 000 of its present age.

8.4 Temperature Fluctuations

The dipole distortion of the cosmic microwave background, shown in Figure 8.2, results from the fact that the universe is not perfectly homogeneous today ($z = 0$). Because we are gravitationally accelerated toward the nearest large lumps of matter, we see a Doppler shift in the radiation of the CMB. The distortions on a smaller angular scale, shown in Figure 8.3, tell us that the universe was not perfectly homogeneous at the time of last scattering ($z \approx 1090$). The angular size of the temperature fluctuations reflects in part the physical size of the density and velocity fluctuations at $z \approx 1090$.

The angular size $\delta\theta$ of a temperature fluctuation in the CMB is related to a physical size ℓ on the last scattering surface by the relation

Small Angle Approx + Angular Size Dist $\Rightarrow \delta\theta = \frac{\ell}{d_A} \Rightarrow$

$$d_A = \frac{\ell}{\delta\theta}, \Leftarrow \begin{cases} (5.36) \\ (6.35) \\ (6.40) \end{cases} \tag{8.46}$$

where d_A is the angular-diameter distance to the last scattering surface. Since the last scattering surface is at a redshift $z_{\rm ls} = 1090 \gg 1$, a good approximation to d_A is given by Equation 6.40:

(6.40) $d_A(z \to \infty) \simeq$

$$d_A \approx \frac{d_{\rm hor}(t_0)}{z_{\rm ls}} \overset{(8.46)}{=} \frac{\ell}{\delta\theta} \tag{8.47}$$

In the Benchmark Model, the current horizon distance is $d_{\rm hor}(t_0) \approx 14\,000$ Mpc, so the angular-diameter distance to the surface of last scattering is

$$\boxed{d_A \approx \frac{14\,000\,\text{Mpc}}{1090} \approx 12.8\,\text{Mpc}.} \quad \text{at } z = 1090 \tag{8.48}$$

Thus, fluctuations on the last scattering surface with an observed angular size $\delta\theta$ had a physical size

(8.46) $\Longrightarrow$

$$\ell = d_A \cdot \delta\theta = 12.8\,\text{Mpc}\left(\frac{\delta\theta}{1\,\text{rad}}\right) = 3.7\,\text{kpc}\left(\frac{\delta\theta}{1\,\text{arcmin}}\right) = 0.22\,\text{Mpc}\left(\frac{\delta\theta}{1\,\text{deg}}\right) \tag{8.49}$$

Local $N(z)$ Survey show peak in power spectrum P_k at $k^{-1} \simeq 128$ Mpc (Great Walls & Voids)

at the time of last scattering. The smallest fluctuations resolved by the *Planck* satellite (Figure 8.3) have an angular size $\delta\theta \approx 5$ arcmin. This corresponds to a physical size of $\ell \approx 18$ kpc at the time of last scattering, or $\ell(1 + z_{\rm ls}) \approx 20$ Mpc today. A sphere with this radius has a baryon mass $M_{\rm bary} \approx 5 \times 10^{13}\,\text{M}_\odot$, about that of a cluster of galaxies. (The Coma cluster has $M_{\rm bary} \sim 2 \times 10^{14}\,\text{M}_\odot$, but it is a very rich cluster.)

Consider the density fluctuations $\delta T/T$ of the cosmic microwave background, as shown in Figure 8.3. Since $\delta T/T$ is defined on the surface of a sphere – the celestial sphere in this case – it is useful to expand it in spherical harmonics:

(8.6) + (8.7) expressed as spherical harmonics:

$$\frac{\delta T}{T}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi), \tag{8.50}$$

where $Y_{lm}(\theta, \phi)$ are the usual spherical harmonic functions. What concerns cosmologists is not the exact pattern of hot spots and cold spots on the sky, but their statistical properties. The most important statistical property of $\delta T/T$ is the correlation function $C(\theta)$. Consider two points on the last scattering surface. Relative to an observer, they are in the directions $\hat{n}$ and $\hat{n}'$, and are separated by an angle θ given by the relation $\cos\theta = \hat{n} \cdot \hat{n}'$. To find the correlation function $C(\theta)$, multiply together the values of $\delta T/T$ at the two points, then average the product over all points separated by the angle θ:

Corr Fn = Fn convolved by itself: "multiplied" but with an offset $\hat{n} \to \hat{n}' \simeq \cos\theta$

$$C(\theta) = \left\langle \frac{\delta T}{T}(\hat{n}) \frac{\delta T}{T}(\hat{n}') \right\rangle_{\hat{n}\cdot\hat{n}' = \cos\theta} \propto \left(\frac{\Delta T}{\langle T \rangle}\right)^2 \Leftrightarrow \text{Power Spectrum} \quad P_k = (F_n)^2 \tag{8.51}$$

Using the expansion of $\delta T/T$ in spherical harmonics, the correlation function can be written in the form

$\Rightarrow$ Corr Fn of $\frac{\delta T}{T}(\theta, \phi)$ = = FT of $P_k(\ell)$

$$C(\theta) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1) C_l P_l(\cos\theta), \tag{8.52}$$

$\theta = \frac{180^\circ}{\ell} = \frac{\pi}{\ell}$ (Rad)

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where P_l are the usual Legendre polynomials:

$$P_0(x) = 1 \quad \text{— monopole} \Longleftrightarrow T_0 = 2.7255\,\text{K}$$
$$P_1(x) = x \quad \text{— dipole} \Longleftrightarrow v_{LSC} \simeq 630\,\text{km s}^{-1} \qquad (\text{“}\infty\text{”}) \tag{8.53}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad \text{— quadrupole}$$

and so forth. In this way, a measured correlation function $C(\theta)$ can be broken down into its multipole moments C_l. The $l = 0$ (monopole) term of the correlation function vanishes if we've defined the mean temperature correctly. The $l = 1$ (dipole) term results primarily from the Doppler shift due to our motion through space. For larger values of l, the term C_l is a measure of temperature fluctuations on an angular scale $\theta \sim 180°/l$. Thus, the multipole l is interchangeable, for all practical purposes, with the angular scale θ. The moments with $l \geq 2$ are of the most interest to astronomers, since they tell us about the fluctuations present at the time of last scattering.

In presenting the results of CMB observations, it is customary to plot the function

$$\Delta_T \equiv \left(\frac{l(l+1)}{2\pi} C_l\right)^{1/2} \langle T \rangle , \tag{8.54}$$

since this function tells us the contribution per logarithmic interval in l to the total temperature fluctuation δT of the cosmic microwave background. Figure 8.6, which shows results from the *Planck* satellite, is a plot of Δ_T as a function of l. The detailed shape of the Δ_T versus l curve contains a wealth of information about the universe at the time of photon decoupling. In the next section we will examine, very briefly, the physics behind the temperature fluctuations, and how we can extract cosmological information from the temperature anisotropy of the cosmic microwave background.

8.5 What Causes the Fluctuations?

At the time of last scattering, a particularly interesting length scale, cosmologically speaking, is the horizon distance,

$$\left.\begin{matrix}(5.51)\\(5.52)\end{matrix}\right\} \Rightarrow \qquad d_{\rm hor}(t_{\rm ls}) = a(t_{\rm ls})c \int_0^{t_{\rm ls}} \frac{dt}{a(t)} . \tag{8.55}$$

Since last scattering takes place long before the cosmological constant plays a significant role in the expansion, we can use the scale factor appropriate for a universe containing just radiation and matter, as given in Equation 5.110. This gives a value for the horizon distance of $\quad t(z = 1090) = 371{,}000\ \text{yrs} \times 3.155\times10^{7}\ \frac{\text{s}}{\text{yr}} \times 2.24c$

$$\frac{c}{1.23\times10^{-18}(1+z)^{3/2}} \simeq \frac{c}{H(z_{ls})} \simeq d_{\rm hor}(t_{\rm ls}) = 2.24ct_{\rm ls} = 0.251\ \text{Mpc}. \tag{8.56}$$

(8.41)

$$\simeq \frac{3.0\times10^{8}\ \text{m/s}}{1.23\times10^{-18}(1091)^{3/2}\ \text{s}^{-1}} \simeq \frac{6.76\times10^{21}\ \text{m}}{1.496\times10^{11}(\text{m/AU})\ 206265(\text{AU/pc})} \simeq 220\ \text{kpc} \simeq 0.22\ \text{Mpc}$$

Peak at $\theta \simeq 0.8^\circ$

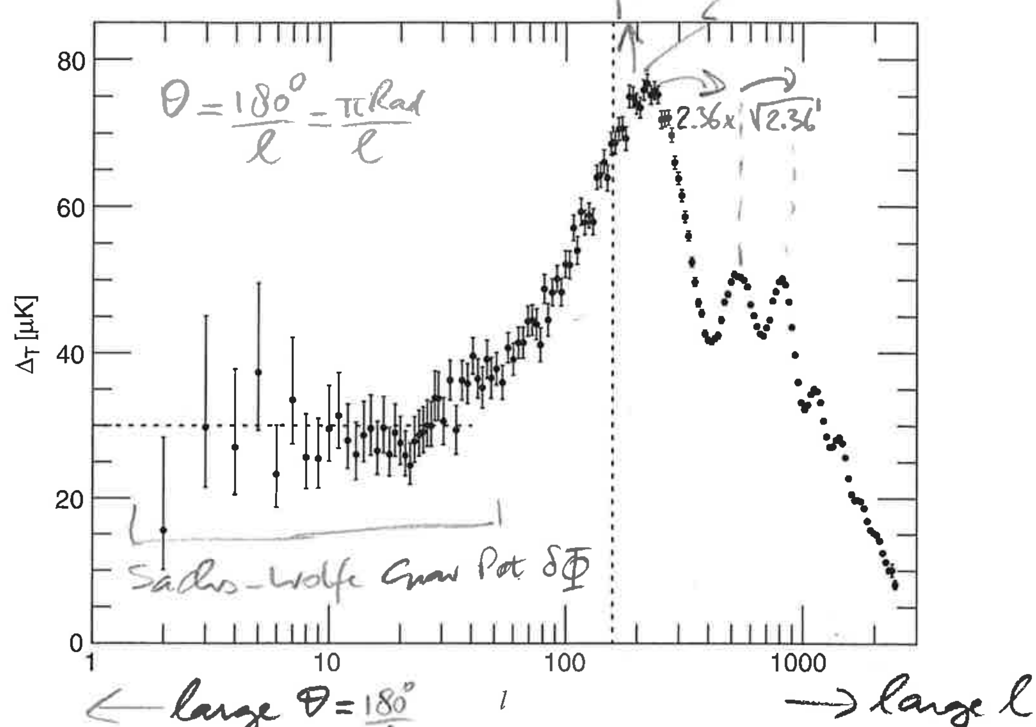

Figure 8.6 Temperature fluctuations Δ_T of the CMB, as observed by *Planck*, expressed as a function of the multipole l. The vertical dotted line shows $l_{\rm hor}$, the multipole corresponding to the horizon size at last scattering. The horizontal dotted line shows the value $\Delta_T \approx 30\,\mu$K at which Δ_T levels off at small l. [data courtesy of *Planck*/ESA]

A patch of the last scattering surface with this physical size will have an angular size, as seen from Earth, of

$$\theta_{\rm hor} = \frac{d_{\rm hor}(t_{\rm ls})}{d_A} = \frac{0.251\,{\rm Mpc}}{12.8\,{\rm Mpc}} \approx 0.020\,{\rm rad} \approx 1.1^\circ. \tag{8.57}$$

This angle corresponds to a multipole $l_{\rm hor} \approx 160$, indicated as the vertical dotted line in Figure 8.6.

The behavior of the Δ_T versus l curve is notably different on large angular scales than on smaller scales. On large angular scales, the value of Δ_T levels off at a nearly constant value $\Delta_T \approx 30\,\mu$K in the limit that $\theta > 4\theta_{\rm hor}$, or $l < 40$. By contrast, on angular scales smaller than $\theta_{\rm hor}$, the Δ_T curve shows a landscape of peaks and valleys rather than a level plateau. The highest peak, at $l \approx 220$, is called the "first peak." The second and third peaks, at $l \approx 520$ and $l \approx 800$, are lower in amplitude. The difference in behavior between the plateau at small l and the peaks and valleys at larger l results from the fact that fluctuations on large angular scales, with $\theta > \theta_{\rm hor}$, have a different physical cause than the fluctuations with $\theta < \theta_{\rm hor}$.

Consider first the large scale fluctuations – those with angular size $\theta > \theta_{\rm hor}$. These temperature fluctuations arise from the gravitational effect of density fluctuations in the distribution of nonbaryonic dark matter. The density of nonbaryonic dark matter at the time of last scattering, since $\varepsilon_{\rm dm} \propto a^{-3} \propto (1+z)^3$, was

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(0.264) (4.30) + (4.31)

$$\varepsilon_{\rm dm}(z_{\rm ls}) = \Omega_{\rm dm,0}\varepsilon_{c,0}(1+z_{\rm ls})^3 \approx 1.7\times10^{12}\,\text{MeV}\,\text{m}^{-3}, \tag{8.58}$$

equivalent to a mass density $\rho \sim 3\times10^{-18}\,\text{kg}\,\text{m}^{-3}$. The density of baryonic matter at the time of last scattering was

$\frac{\varepsilon_{dm}}{\varepsilon_{bar}} \approx 5.5\times$!

(4.31) $\varepsilon_{c,0} = \frac{3c^2H_0^2}{8\pi G} \Rightarrow$ (0.048)

$$\varepsilon_{\rm bary}(z_{\rm ls}) = \Omega_{\rm bary,0}\varepsilon_{c,0}(1+z_{\rm ls})^3 \approx 3.1\times10^{11}\,\text{MeV}\,\text{m}^{-3}. \tag{8.59}$$

$= 4870\ \frac{\text{MeV}}{\text{m}^3} = 1.28\times10^{11}\ \frac{M_\odot}{\text{Mpc}^3}$

The density of photons at the time of last scattering, since $\varepsilon_\gamma \propto a^{-4} \propto (1+z)^4$, was

$\approx 5.35\times10^{-5}$ (4.31) (5.13) $\qquad \frac{\varepsilon_{bar}}{\varepsilon_{phot}} \approx 0.8 \approx \frac{1}{1.26}$

$$\varepsilon_\gamma(z_{\rm ls}) = \Omega_{\gamma,0}\varepsilon_{c,0}(1+z_{\rm ls})^4 \approx 3.9\times10^{11}\,\text{MeV}\,\text{m}^{-3}. \tag{8.60}$$

Thus, at the time of last scattering, $\varepsilon_{\rm dm} > \varepsilon_\gamma > \varepsilon_{\rm bary}$, with dark matter, photons, and baryons having energy densities in the ratio 5.5 : 1.24 : 1. The nonbaryonic dark matter dominated the energy density ε, and hence the gravitational potential, of the universe at the time of last scattering.

Suppose that the density of the nonbaryonic dark matter at the time of last scattering was not perfectly homogeneous, but varied as a function of position. Then we could write the energy density of the dark matter as

$$\varepsilon(\vec{r}) = \bar{\varepsilon} + \delta\varepsilon(\vec{r}), \tag{8.61}$$

where $\bar{\varepsilon}$ is the spatially averaged energy density of the nonbaryonic dark matter, and $\delta\varepsilon$ is the local deviation from the mean. In the Newtonian approximation, the spatially varying component of the energy density, $\delta\varepsilon$, gives rise to a spatially varying gravitational potential $\delta\Phi$. The link between $\delta\varepsilon$ and $\delta\Phi$ is Poisson's equation:

$$\boxed{\nabla^2(\delta\Phi) = \frac{4\pi G}{c^2}\delta\varepsilon.} \tag{8.62}$$

Unless the distribution of dark matter were perfectly smooth at the time of last scattering, the fluctuations in its density would necessarily have given rise to fluctuations in the gravitational potential.

Consider the fate of a CMB photon that happens to be at a local minimum of the potential at the time of last scattering. (Minima in the gravitational potential are known colloquially as "potential wells.") In climbing out of the potential well, it loses energy, and consequently is redshifted. Conversely, a photon that happens to be at a potential maximum when the universe became transparent gains energy as it falls down the "potential hill," and thus is blueshifted. When we look at the last scattering surface on large angular scales, cool (redshifted) spots correspond to minima in $\delta\Phi$; hot (blueshifted) spots correspond to maxima in $\delta\Phi$. A detailed general relativistic calculation, first performed by Sachs and Wolfe in 1967, tells us that

$$\boxed{\frac{\delta T}{T} = \frac{1}{3}\frac{\delta\Phi}{c^2}.} \tag{8.63}$$

Thus, the temperature fluctuations on large angular scales ($\theta > \theta_{\rm hor} \approx 1.1°$) give us a map of the potential fluctuations $\delta\Phi$ present at the time of last scattering. In particular, the fact that the observed moments Δ_T are constant over a wide range of angular scales, from $l \sim 2$ to $l \sim 40$, tells us that the potential fluctuations $\delta\Phi$ were constant over a wide range of physical scales. The creation of temperature fluctuations by variations in the gravitational potential is known generally as the *Sachs–Wolfe effect*, and the region of the Δ_T curve where the temperature fluctuations are nearly constant ($l < 40$) is known as the Sachs–Wolfe plateau.

(8.58)-(8.60)

$\frac{\varepsilon_\gamma + \varepsilon_b}{\varepsilon_{dm}} \simeq \frac{7\times10^{11}}{1.7\times10^{12}} \simeq 0.41$

On smaller scales ($\theta < \theta_{\rm hor}$), the origin of the temperature fluctuations in the CMB is complicated by the behavior of the photons and baryons. Consider the situation immediately prior to photon decoupling. The photons, electrons, and protons together make a single photon-baryon fluid, whose energy density is only 40 percent that of the dark matter. Thus, the photon-baryon fluid moves primarily under the gravitational influence of the dark matter, rather than under its own self-gravity. If the photon-baryon fluid finds itself in a potential well of the dark matter, it will start to fall toward the center of the well. However, as the photon-baryon fluid is compressed by gravity, its pressure starts to rise along with its increasing density. Eventually, the pressure is sufficient to cause the fluid to expand outward. As the expansion continues, the pressure drops until gravity causes the photon-baryon fluid to fall inward again. As the cycle continues, the inward and outward oscillations of the photon-baryon fluid are called *acoustic oscillations*, since they represent a standing sound wave.

$\varepsilon_\gamma \propto T^4$ ⇔

COSMIC STEPHAN BOLTZMANN (2.28)+(2.29)

If the photon-baryon fluid within a potential well is at maximum compression at the time of photon decoupling, its density will be higher than average, and since $T \propto \varepsilon_\gamma^{1/4}$, the liberated photons will be hotter than average. Conversely, if the photon-baryon fluid within a potential well is at maximum expansion at the time of decoupling, the liberated photons will be slightly cooler than average. If the photon-baryon fluid is in the process of expanding or contracting at the time of decoupling, the Doppler effect will cause the liberated photons to be cooler or hotter than average, depending on whether the photon-baryon fluid was moving away from our location or toward it at the time of photon decoupling. Computing the exact shape of the Δ_T versus l curve expected in a particular model universe is a rather complicated chore. Generally speaking, however, the first peak in the Δ_T curve (at $l \approx 220$ or $\theta \approx 0.8°$) represents the potential wells within which the photon-baryon fluid had just reached maximum compression at the time of last scattering. These potential wells have a size comparable to the *sound horizon distance* for the photon-baryon fluid at the time of last scattering. The sound horizon distance d_s is the maximum proper distance that a sound wave in the photon-baryon fluid can have traveled since the Big Bang. By analogy with the usual horizon distance (Equation 8.55), the sound horizon distance at the time of last scattering is

(8.55) but for sound horizon $\Rightarrow$

$$d_s(t_{\rm ls}) = a(t_{\rm ls}) \int_0^{t_{\rm ls}} \frac{c_s(t)dt}{a(t)}. \tag{8.64}$$

For most of the time prior to last scattering, the baryons were an insignificant contaminant in the photon-baryon fluid. We can thus make the approximation that the sound speed in the photon-baryon fluid was $c_s \approx c/\sqrt{3}$, the same as that of a pure photon gas. With this approximation, the sound horizon distance at last scattering was

(8.56)

(8.56)

$$d_s(t_{\rm ls}) \approx \frac{1}{\sqrt{3}} d_{\rm hor}(t_{\rm ls}) \approx 0.145\,{\rm Mpc} = \frac{1}{\sqrt{3}}\, 0.251\,{\rm Mpc} \tag{8.65}$$

Hot spots with this physical size have an angular size, as viewed by us today, of

$$\theta_s \approx \frac{d_s(t_{\rm ls})}{d_A} \approx \frac{0.145\,{\rm Mpc}}{12.8\,{\rm Mpc}} \approx 0.011\,{\rm rad} \approx 0.7^\circ. \tag{8.66}$$

The location and the amplitude of the first peak in Figure 8.6 provide very useful cosmological information. The angular size $\theta_{\rm peak}$ at which the peak occurs must be similar to the angle θ_s subtended by the sound horizon distance. However, the estimate for θ_s given in Equation 8.66 assumes a flat universe. In a negatively curved universe ($\kappa = -1$), the angular size θ of an object of known physical size at a known redshift is *smaller* than it is in a positively curved universe ($\kappa = +1$). Thus, if the universe were negatively curved, the first peak in Δ_T would be shifted to a smaller angle; if the universe were positively curved, the peak would be shifted to a larger angle. The observed position of the first acoustic peak is consistent with $\kappa = 0$, or $\Omega_0 = 1$. Figure 8.7 shows the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ permitted by the present CMB data. Note that the contour that shows the best fit to the CMB data (the black ellipse) is roughly perpendicular to the contour permitted by the type Ia supernova results (the gray ellipse). The small overlap region between the two results tells us that a spatially flat universe, with $\Omega_{m,0} \sim 0.3$ and $\Omega_{\Lambda,0} \sim 0.7$, agrees with both the CMB results and the supernova results.

Planck 2016
$\Omega_{m,0} \simeq 0.315$
$\Omega_{\Lambda,0} \simeq 0.685$
$H_0 \simeq 67 \pm 1$

The *amplitude* of the first peak in the Δ_T versus l plot also yields useful cosmological knowledge. The amplitude is quite sensitive to the sound speed c_s of the photon-baryon fluid, with a lower speed giving a higher amplitude. Since the sound speed is $c_s = \sqrt{w_{\rm pb}}\, c$, and the equation-of-state parameter $w_{\rm pb}$ is in turn dependent on the baryon-to-photon ratio, the amplitude of the peak is a useful diagnostic of the baryon density of the universe. Detailed analysis of the Δ_T curve yields a baryon-to-photon ratio

$$\eta = (6.10 \pm 0.06) \times 10^{-10}. \quad (\eta^{-1} \simeq 1.64 \times 10^9) \tag{8.67}$$

This value for η can be converted into a value for the current baryon density by the relation

$$n_{\rm bary,0} = \eta n_{\gamma,0} = 0.251 \pm 0.003\,{\rm m}^{-3}. \tag{8.68}$$

(2.35) $\Rightarrow$ $n_{\gamma,0} = 4.107 \times 10^8\,{\rm m}^{-3}$

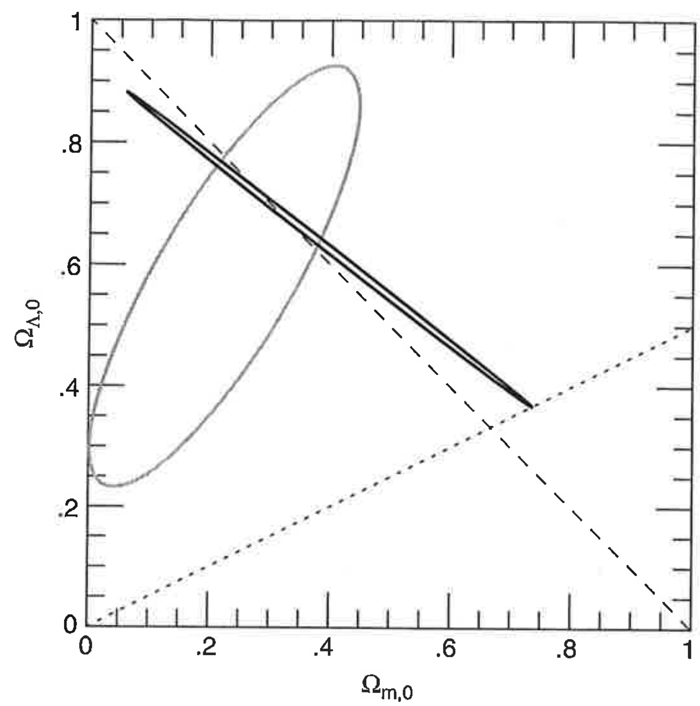

Figure 8.7 The elongated black ellipse represents the 95% confidence interval for the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ that best fit the *Planck* CMB data. For comparison, the gray ellipse shows the 95% confidence interval from the supernova data, repeated from Figure 6.6. [Anže Slosar & José Alberto Vázquez, BNL]

936 MeV (8.4)

Since the majority of baryons are protons, we may write, to acceptable accuracy,

$$\varepsilon_{\mathrm{bary},0} = (m_p c^2) n_{\mathrm{bary},0} = 235 \pm 3\,\mathrm{MeV\,m^{-3}}. \tag{8.69}$$

This translates into a density parameter for baryons of

$$\boxed{\Omega_{\mathrm{bary},0} = \frac{\varepsilon_{\mathrm{bary},0}}{\varepsilon_{c,0}} = 0.048 \pm 0.003 = \Omega_b} \tag{8.70}$$

with most of the uncertainty in $\Omega_{\mathrm{bary},0}$ coming from the uncertainty in the critical density $\varepsilon_{c,0}$.

$\varepsilon_{c,0} = 4870 \frac{\mathrm{MeV}}{m^3}$ (4.31)

Exercises

8.1 The purpose of this problem is to determine how changing the value of the baryon-to-photon ratio, η, affects the recombination temperature in the early universe. Plot the fractional ionization X as a function of temperature, in the range $3000\,\mathrm{K} < T < 4500\,\mathrm{K}$; first make the plot assuming $\eta = 4 \times 10^{-10}$, then assuming $\eta = 8 \times 10^{-10}$. How much does this change in η affect the computed value of the recombination temperature T_{rec}, if we define T_{rec} as the temperature at which $X = \frac{1}{2}$?