Dark Matter

Cosmologists, over the years, have dedicated much time and effort to determining the matter density of the universe. There are many reasons for this obsession. First, the density parameter in matter, $\Omega_{m,0}$, is important in determining the spatial curvature and expansion rate of the universe. Even if the cosmological constant is nonzero, the matter content of the universe is not negligible today, and was the dominant component in the fairly recent past. Another reason for wanting to know the matter density of the universe is to find out what the universe is made of. What fraction of the density is made of stars, and other familiar types of baryonic matter? What fraction of the density is made of dark matter? What constitutes the dark matter – cold stellar remnants, black holes, exotic elementary particles, or some other substance too dim for us to see? These questions, and others, have driven astronomers to take a census of the universe, to find out what types of matter it contains, and in what quantities.

We have already seen in the previous chapter one method of putting limits on $\Omega_{m,0}$. The apparent magnitude (or flux) of type Ia supernovae as a function of redshift is consistent with a flat universe having $\Omega_{m,0} \approx 0.3$ and $\Omega_{\Lambda,0} \approx 0.7$. However, neither $\Omega_{m,0}$ nor $\Omega_{\Lambda,0}$ is individually well-constrained by the supernova observations. The supernova data are consistent with $\Omega_{m,0} = 0$ if $\Omega_{\Lambda,0} \approx 0.3$; they are also consistent with $\Omega_{m,0} = 0.45$ if $\Omega_{\Lambda,0} \approx 0.9$. In order to determine $\Omega_{m,0}$ more accurately, we will have to adopt alternate methods of estimating the matter content of the universe.

Planck 2016 + SN data etc: $\Omega_{m,0} \simeq 0.32$ $\Omega_{\Lambda,0} \simeq 0.68$ ± 0.02

7.1 Visible Matter

Some types of matter, such as stars, help astronomers to detect them by broadcasting photons in all directions. Stars emit light primarily in the infrared, visible, and ultraviolet range of the electromagnetic spectrum. Suppose, for instance, you install a V-band filter on your telescope. Such a filter allows photons in the wavelength range 500 nm $< \lambda <$ 590 nm to pass through. The "V" in V-band stands for "visual"; although your eyes can detect the broader wavelength range 400 nm $< \lambda <$ 700 nm, a V-band filter lets through the green and yellow wavelengths of light to which your retina is most sensitive. About 12 percent of the Sun's luminosity can pass through a V-band filter; thus, the Sun's luminosity in the V band is $L_{\odot,V} \approx 0.12 L_\odot \approx 4.6 \times 10^{25}$ watts.[1]

Surveys of galaxies reveal that in the local universe (out to $d \sim 0.1c/H_0$), the luminosity density in the V band is

$$\rho \cdot L = nL = \Psi_V = 1.1 \times 10^8 L_{\odot,V}\,\mathrm{Mpc}^{-3}. \tag{7.1}$$

To convert a luminosity density into a mass density $\rho_\star$ of stars, we need to know the *mass-to-light ratio* of the stars. If all stars were identical to the Sun, we could simply say that there is one solar mass of stars for each solar luminosity of output power, or $\langle M/L_V \rangle = 1\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$; this corresponds to about 43 metric tons for every watt of yellow-green light. However, stars are not uniform in their properties.

Sun has $\frac{M}{L} = \frac{1 M_\odot}{1 L_\odot} = 1$!

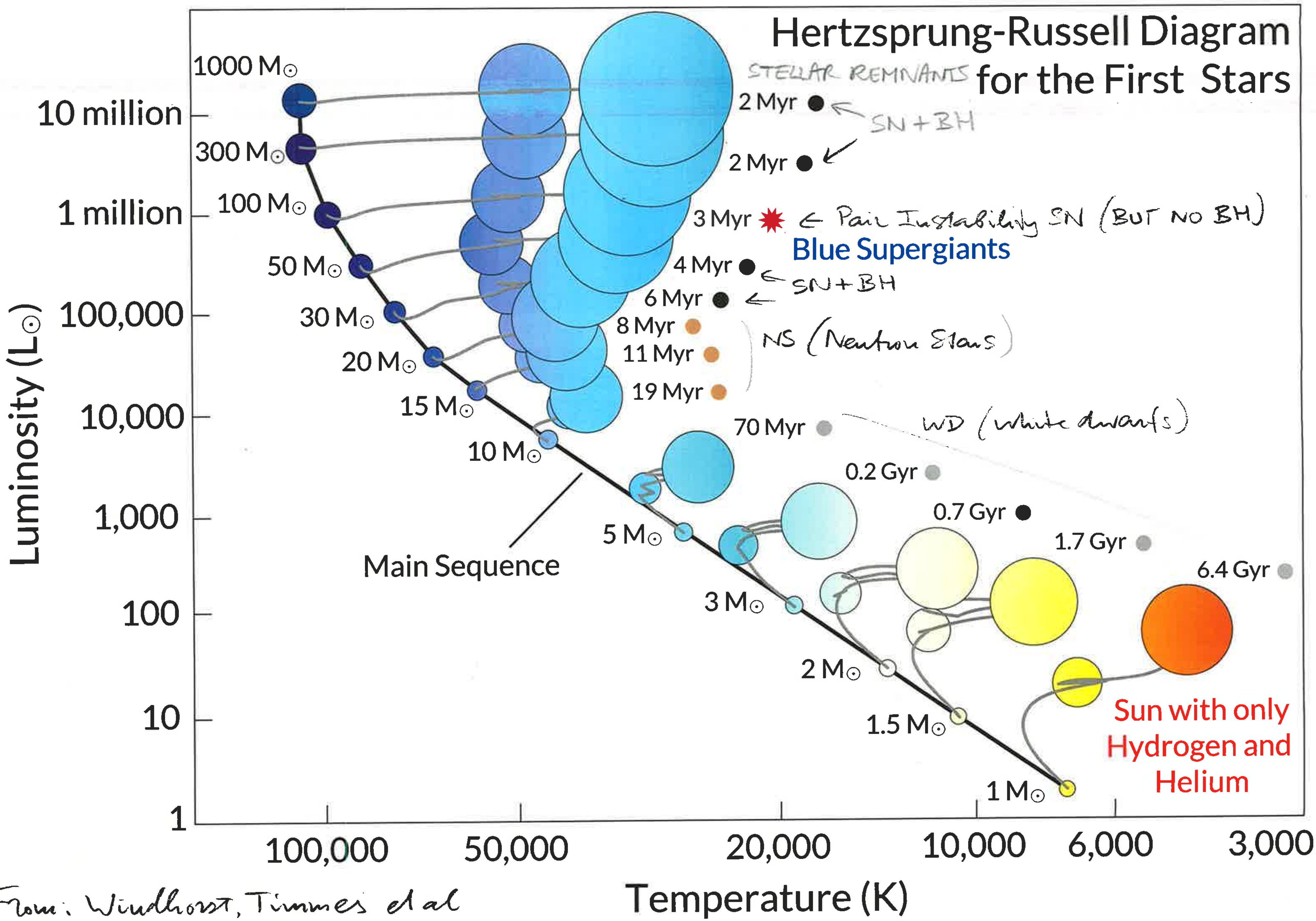
Consider, for instance, the stars that astronomers refer to as "main sequence" stars; these are stars that are powered, like the Sun, by hydrogen fusion in their cores. The surface temperature and luminosity of a main sequence star are determined by its mass, with the most massive stars being the hottest and brightest. Astronomers find it useful to encode the surface temperature of a star as a letter, called the *spectral type* of the star. For historical reasons, these spectral types are not in alphabetical order: from hottest to coolest, they are O, B, A, F, G, K, and M. (Although the sequence of spectral types looks like an explosion in an alphabet soup factory, it does provide us with a useful shorthand: hot, luminous, massive main sequence stars can be called "O stars" for short, while cool, dim, low-mass main sequence stars are "M stars.") An O star with mass $M = 60\,\mathrm{M}_\odot$ has a V-band luminosity $L_V \approx 20\,000\,\mathrm{L}_{\odot,V}$, and thus a mass-to-light ratio $M/L_V \approx 0.003\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$. By contrast, an M star with mass $M = 0.1\,\mathrm{M}_\odot$ has $L_V \approx 5 \times 10^{-5}\,\mathrm{L}_{\odot,V}$, and thus a mass-to-light ratio $M/L_V \approx 2000\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$.

M-L RELATION (for $M \lesssim 30 M_\odot$)
$L \propto M^{3.5}$
$\Rightarrow \frac{M}{L} \propto \frac{M}{M^{3.5}} \propto M^{-2.5}$
$\Rightarrow \tau_{MS} \propto M^{-2.5}$
A $10 M_\odot$ star lives 300× shorter than Sun (see Fig 2 of Windhorst et al 2018, ApJS, 234, 14) using $L \propto \frac{Mc^2}{\tau}$

The mass-to-light ratio of the stars in a galaxy will therefore depend on the mix of stars that it contains. The physical processes that form stars are found empirically to favor low-mass stars over high-mass stars. In a star-forming region, the *initial mass function* $\chi(M)$ is defined so that $\chi(M)dM$ is the number of stars created with masses in the range $M \to M + dM$. At masses $M > 1\,\mathrm{M}_\odot$, the initial mass function is well fitted by a power law,

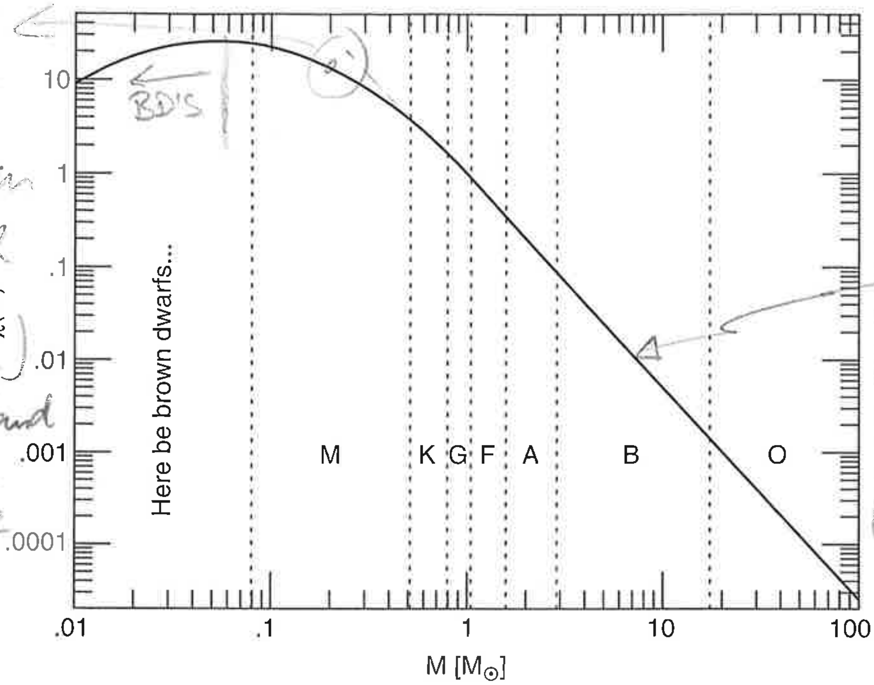
$$\boxed{\chi(M) \propto M^{-\beta}} \qquad [M > 1\,\mathrm{M}_\odot]. \quad \text{with } \beta \simeq 2.3 \tag{7.2}$$

[1] Although old-fashioned incandescent light bulbs are castigated for their inefficiency at producing visible light, the Sun isn't hyper-efficient at producing visible light either. (Or rather, to get the causality right, our eyes haven't evolved to be hyper-efficient at detecting sunlight.) About 10% of the Sun's luminosity is in the ultraviolet range and 50% is in the infrared, leaving only 40% in the wavelength range $\lambda = 400 \to 700$ nm.



From: Windhorst, Timmes et al
2018 ApJ Suppl. 234, 41.

$M_c \simeq 0.2 M_\odot \Leftarrow$ is where the transition occurs between 1) Log-normal (downward log-parabola) for $M \ll M_c$, and 2) Power-law for $M \gg M_c \simeq 0.2$



Salpeter Initial Mass Function: $\frac{dN}{dM} \propto M^{-2.35} = M^{-\beta}$

More general, for $M \gtrsim 1 M_\odot$: $\beta \simeq 2.0$ (see Windhorst et al 2018 ApJS, 234, 41).

Figure 7.1 An initial mass function for stars and brown dwarfs. Mass ranges corresponding to the standard stellar spectral types O through M are indicated, as well as the low-mass realm of brown dwarfs. The values $\beta = 2.3$, $M_c = 0.2\,M_\odot$ and $\sigma = 0.5$ are assumed in the Chabrier mass function of Equations 7.2 and 7.3.

Power-law: $\chi(M) \propto M^{-\beta}$ for $M > 1 M_\odot$ ($\beta = 2.3$ to 2.0 or 1.5 for Pop III) (7.2)

The power-law index β varies from location to location, but a value $\beta = 2.3$ is typical. At lower masses, the shape of the initial mass function is less well determined, but a log-normal distribution is found to give a reasonable fit:

"log-normal" $\Rightarrow$
$$\chi(M) \propto \frac{1}{M} \exp\left(-\frac{(\log M - \log M_c)^2}{2\sigma^2}\right) \qquad [M < 1\,M_\odot]. \qquad (7.3)$$

The characteristic mass M_c and the width σ of the distribution vary from location to location. However, typical values, when masses are measured in units of the solar mass, are $M_c \approx 0.2$ and $\sigma \approx 0.5$ dex

The initial mass function found by combining Equations 7.2 and 7.3 is plotted in Figure 7.1.[2] Gaseous spheres less massive than $M = 0.08\,M_\odot$ are actually *brown dwarfs* rather than stars. The difference between a brown dwarf and a star is that a brown dwarf is too low in mass for hydrogen fusion to be ignited at its center. Since brown dwarfs are not powered by nuclear fusion, they tend to be even cooler and dimmer than M stars. The initial mass function for stars and brown dwarfs is highest in the mass range $0.02\,M_\odot \rightarrow 0.2\,M_\odot$; O stars, with $M > 18\,M_\odot$, are far out on the power-law tail of the initial mass function. At the time of formation, there will be about 250 low-mass M stars for every O star.

$250 \langle M_M \rangle \simeq 250 \times 0.1 \simeq \langle M_O \rangle \simeq 25\text{–}50\, M_\odot$

[2] An initial mass function that takes the form of a log-normal distribution with a power-law tail to high masses is called a Chabrier function, after the astronomer Gilles Chabrier.

Although the total mass of 250 M stars is comparable to that of a single O star, their total V-band luminosity is negligible compared to the O star's luminosity. In galaxies actively forming stars today, the mass-to-light ratio of the stellar population is found to be as small as $M/L_V \approx 0.3\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$.

Although O stars are extremely luminous, they are also short-lived. An O star with a mass $M = 60\,\mathrm{M}_\odot$ will run out of fuel for fusion in a time $t \approx 3$ Myr; it will then explode as a type II supernova. Thus, a galaxy that is quiescent (that is, one that has long since stopped forming new stars) will lack O stars. The mass-to-light ratio of quiescent galaxies can rise to as large as $M/L_V \approx 8\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$. In the local universe, there is a mix of star-forming and quiescent galaxies, so we can't go too badly wrong if we take an averaged mass-to-light ratio of $\langle M/L_V\rangle \approx 4\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$. With this value, we find that the mass density of stars in the universe today is

(7.1)
$\Psi_V \simeq 1.1\times10^8\ L_\odot\,\mathrm{Mpc}^{-3} \Longrightarrow$

$$\rho_{\star,0} = \underbrace{\langle M/L_V\rangle}_{\sim 4}\Psi_V \approx \overbrace{4\times10^8}^{4.4\times10^8}\,\mathrm{M}_\odot\,\mathrm{Mpc}^{-3}. \tag{7.4}$$

Since the current critical density of the universe, expressed as a mass density, is $\rho_{c,0} = 1.28\times10^{11}\,\mathrm{M}_\odot\,\mathrm{Mpc}^{-3}$, the current density parameter of stars is

(4.32)
$\rho_{c,0} \equiv \frac{\varepsilon_{c,0}}{c^2} = \frac{3c^2H_0^2}{8\pi G\,c^2} = 1.28\times10^{11}\ M_\odot\,\mathrm{Mpc}^{-3} \rightarrow$

$$\Omega_{\star,0} = \frac{\rho_{\star,0}}{\rho_{c,0}} \approx \frac{\overset{4.4}{4}\times10^8\,\mathrm{M}_\odot\,\mathrm{Mpc}^{-3}}{1.28\times10^{11}\,\mathrm{M}_\odot\,\mathrm{Mpc}^{-3}} \approx 0.003\overset{4}{} \approx 0.34\%\ \text{of}\ \Omega_{crit} \tag{7.5}$$

Obs/Dist	M/L $M_\odot/L_\odot$
STARS	
100 $M_\odot$/kpc	$\sim 10^{-4}$
10 $M_\odot$/10pc	3×10^{-4}
1 $M_\odot$/1 AU	$\equiv 1.0$
0.1 $M_\odot$/pc	$\sim 10^3$
Gal/★'s only	
SFG/4 Mpc	~ 0.3
⟨SP⟩/10Mpc	~ 4
Ell/15Mpc	~ 8
Gals: ★+DM	
Virgo/15Mpc	$\sim 10^2$
Coma/100Mpc	~ 400
Univ/4Gpc	~ 300

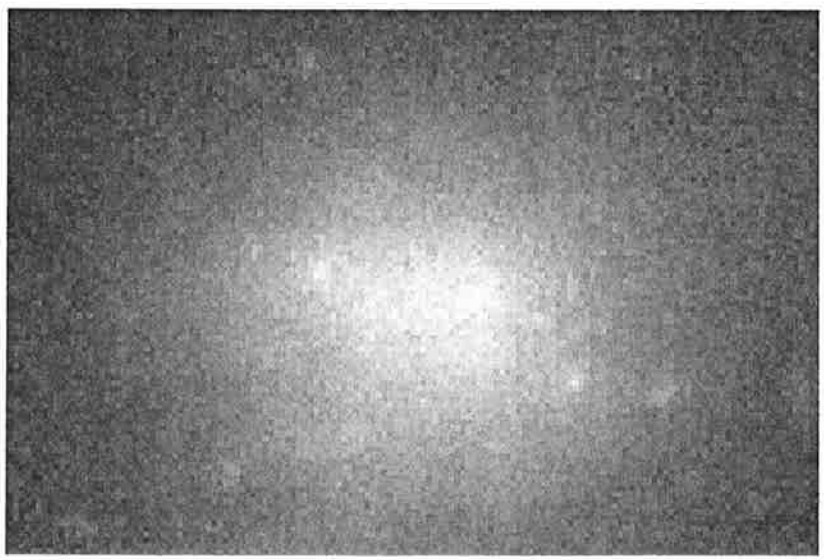
By this accounting, stars make up just 0.3% of the density needed to flatten the universe. The density parameter in stars is boosted slightly if you broaden the category of stars to include stellar remnants such as white dwarfs, neutron stars, and black holes, as well as substellar objects such as brown dwarfs. However, even when you add ex-stars and not-quite-stars to the total, you still find a density parameter $\Omega_{\star,0} < 0.005$.

Galaxies also contain baryonic matter that is not in the form of stars, stellar remnants, or brown dwarfs. In our galaxy and in M31, for instance, the mass of interstellar gas is about 20 percent of the mass in stars. In irregular galaxies such as the Magellanic Clouds, the ratio of gas to stars is even higher. In addition, there is a significant amount of gas between galaxies. Consider a rich cluster of galaxies such as the Coma cluster, located about 100 Mpc from our galaxy, in the direction of the constellation Coma Berenices. At visible wavelengths, as shown in Figure 7.2, most of the light comes from the stars within the cluster's galaxies. The two brightest galaxies in the Coma cluster, NGC 4889 (on the left in Figure 7.2) and NGC 4874 (on the right), each have a luminosity $L_V \approx 2.5\times10^{11}L_{\odot,V}$.[3] The Coma cluster contains thousands of galaxies, most of them far less luminous than NGC 4889 and NGC 4874; their summed luminosity in the V band comes to $L_{\mathrm{Coma},V} \approx 5\times10^{12}\,\mathrm{L}_{\odot,V}$. If the mass-to-light ratio of the stars in the Coma

[3] The bright star (with diffraction spikes) just above NGC 4874 in Figure 7.2 is HD 112887, a main sequence F star at a distance $d \approx 77$ pc, less than a millionth the distance to the Coma cluster. Almost every other light source in Figure 7.2 is a galaxy within the Coma cluster.

Virial

Figure 7.2 The Coma cluster as seen in visible light. The region shown is 36 arcminutes by 24 arcminutes, equivalent to 1.1 Mpc by 0.7 Mpc at the distance of the Coma cluster. [Sloan Digital Sky Survey]



X-Ray
$\Downarrow$
$T \approx 10^8$ K

Figure 7.3 The Coma cluster as seen in X-ray light. The location, orientation, and scale are the same as in the visible light image of Figure 7.2. [NASA SkyView: data from *ROSAT* orbiting X-ray observatory]

cluster is $\langle M/L_V \rangle \approx 4\,M_\odot/L_{\odot,V}$, then the total mass of stars in the Coma cluster is $M_{\text{Coma},\star} \approx 2 \times 10^{13}\,M_\odot$. Although 20 trillion solar masses represents a lot of stars, the stellar mass in the Coma cluster is small compared to the mass of the hot, intracluster gas between the galaxies in the cluster. X-ray images, such as the one shown in Figure 7.3, reveal that hot, low-density gas, with a typical temperature of $T \approx 10^8$ K, fills the space between clusters, emitting X-rays with a typical energy of $E \sim kT_{\text{gas}} \sim 9$ keV. The total amount of X-ray emitting gas in

$\boxed{\text{H/w 7.3}}$ Compute v if $\frac{1}{2} m_p v^2 = kT_g$ with $T_g \approx 10^8$ K or ~ 10 keV
Discuss the value of v you get for a cluster (in km/s)

Coma

$M_{gas} \simeq 10 M_{*'s}$

the Coma cluster is estimated to be $M_{\text{Coma,gas}} \approx 2 \times 10^{14}\,\text{M}_\odot$, roughly ten times the mass in stars.

+Clusters occupy small fraction of the total cosmological volume

Not all the baryonic matter in the universe is easy to detect. About 85% of the baryons in the universe are in the extremely tenuous gas of intergalactic space, outside galaxies and clusters of galaxies. Much of this intergalactic gas is too low in density to be readily detected with current technology. The best limits on the baryon density of the universe actually come from observations of the cosmic microwave background and from the predictions of primordial nucleosynthesis in the early universe. The cosmic microwave background has temperature fluctuations whose properties depend on the baryon-to-photon ratio when the universe was a quarter of a million years old. In addition, the efficiency with which nucleosynthesis takes place in the early universe, converting hydrogen into deuterium, helium, lithium, and other elements, depends on the baryon-to-photon ratio when the universe was a few minutes old. Both these sources of information about the early universe indicate that the density parameter of baryonic matter today must be

(7.5)

$$\frac{\Omega_{baryon}}{\Omega_{gas+*'s}} \simeq 6 \Leftarrow \frac{\Omega_{bar}}{\Omega_{*,0}} \simeq 14 \Leftarrow \Omega_{\text{bary},0} = 0.048 \pm 0.003 \simeq 14\,\Omega_{*,0} \tag{7.6}$$

ten to twenty times the density parameter for stars. When you stare up at the night sky and marvel at the glory of the stars, you are actually marveling at a minority of the baryonic matter in the universe.

7.2 Dark Matter in Galaxies

The situation, in fact, is even more extreme than stated in the previous section. Not only is most of the baryonic matter undetectable by our eyes, but most of the matter is not even baryonic. The majority of the matter in the universe is *nonbaryonic dark matter*, which doesn't absorb, emit, or scatter light of any wavelength. One way of detecting dark matter is to look for its gravitational influence on visible matter. A classic method of detecting dark matter involves looking at the orbital speeds of stars in spiral galaxies such as our own galaxy and M31. Spiral galaxies contain flattened disks of stars; within the disk, stars are on nearly circular orbits around the center of the galaxy. The Sun, for instance, is on such an orbit – it is $R = 8.2\,\text{kpc}$ from the galactic center, and has an orbital speed of $v = 235\,\text{km s}^{-1}$.

Suppose that a star is on a circular orbit around the center of its galaxy. If the radius of the orbit is R and the orbital speed is v, then the star experiences an acceleration (Newton)

$$F = m \times a = \frac{v^2}{R}, \times m = F_{cent} = \frac{mv^2}{R} \tag{7.7}$$

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$M = m_{test} = M_\odot$!

directed toward the center of the galaxy. If the acceleration is provided by the gravitational attraction of the galaxy, then

(Newton) (7.7)

$$F = m \times a = \frac{GM(R)}{R^2} \times m = F_{cent} = \frac{m v^2}{R} \qquad (7.8)$$

where $M(R)$ is the mass contained within a sphere of radius R centered on the galactic center.[4] The relation between v and M is found by setting Equation 7.7 equal to Equation 7.8:

$$\frac{v^2}{R} = \frac{GM(R)}{R^2}, \qquad (7.9)$$

or

Kepler's $v(R)$!

$$v = \sqrt{\frac{GM(R)}{R}} \propto \frac{1}{\sqrt{R}} \qquad (7.10)$$

NOT WHAT IS OBSERVED FOR OUR OR ANY GALAXY

The surface brightness I of the disk of a spiral galaxy typically falls off exponentially with distance from the center:

$R_s \simeq$ few kpc

$$\boxed{I(R) = I(0) \exp\left(-\frac{R}{R_s}\right),} \qquad (7.11)$$

with the scale length R_s typically being a few kiloparsecs. For our galaxy, the scale length measured in the V band is $R_s \approx 4\,\mathrm{kpc}$; for M31, a somewhat larger disk galaxy, $R_s \approx 6\,\mathrm{kpc}$. Once you are a few scale lengths from the center of the spiral galaxy, the mass of stars inside R becomes essentially constant. Thus, if stars contributed all, or most, of the mass in a galaxy, the velocity would fall as $v \propto 1/\sqrt{R}$ * at large radii. This relation between orbital speed and orbital radius, $v \propto 1/\sqrt{R}$, is referred to as "Keplerian rotation," since it's what Kepler found for orbits in the solar system, where 99.8 percent of the mass is contained within the Sun.

$M_\odot \simeq 3.33 \times 10^5 M_\oplus$
$M_{♃} \simeq 320\, M_\oplus$
$M_\odot \simeq 1050\, M_{♃}$
$M_{SS} \simeq 1.002\, M_\odot$

The first astronomer to detect the rotation of M31 was Vesto Slipher, in 1914, two years after he measured the blueshift resulting from its motion toward our own galaxy. However, given the difficulty of measuring the spectra at low surface brightness, the orbital speed v at $R > 3R_s = 18\,\mathrm{kpc}$ was not accurately measured until more than half a century later. In 1970, Vera Rubin and Kent Ford looked at emission lines from regions of hot ionized gas in M31, and were able to find the orbital speed $v(R)$ out to a radius $R = 24\,\mathrm{kpc} = 4R_s$. Their results gave no sign of a Keplerian decrease in the orbital speed. At $R > 4R_s$, a small amount of atomic hydrogen is still in the disk of M31, which can be detected by means of its emission line at $\lambda = 21\,\mathrm{cm}$. From observations of the Doppler shift of this emission line, the orbital speed is found to be nearly constant at $v(R) \approx 230\,\mathrm{km\,s^{-1}}$ out to $R = 35\,\mathrm{kpc} \approx 6R_s$. Since the orbital speed of the stars and

$M(R) \simeq$ $\simeq$ CONSTANT !!

* [4] Equation 7.8 assumes that the mass distribution of the galaxy is spherically symmetric. This is not, strictly speaking, true (the stars in the disk obviously have a flattened distribution), but the flattening of the galaxy provides only a small correction to the equation for the gravitational acceleration. Brushed under π:

Inclination: $\frac{b}{a} = \cos i \Rightarrow \sin i = \sqrt{1 - (b/a)^2}$ since $\cos^2 i + \sin^2 i \equiv 1$

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where v_{gal} is the radial velocity of the galaxy as a whole, resulting from the expansion of the universe, and $v(R)$ is the orbital speed at a distance R from the center of the disk. We can thus compute the orbital speed $v(R)$ in terms of observable properties as

$$v(R) = \frac{v_r(R) - v_{\text{gal}}}{\sin i} = \frac{v_r(R) - v_{\text{gal}}}{\sqrt{1 - b^2/a^2}}. \tag{8.13}$$

The first astronomer to detect the rotation of M31 was Vesto Slipher, in 1914. However, given the difficulty of measuring the spectra at low surface brightness, the orbital speed v at $R > 3R_s = 18\,\text{kpc}$ was not accurately measured until more than half a century later. In 1970, Vera Rubin and Kent Ford looked at emission lines from regions of hot ionized gas in M31, and were able to find the orbital speed $v(R)$ out to a radius $R = 24\,\text{kpc} = 4R_s$. Their results, shown as the open circles in Figure 8.4, give no sign of a Keplerian decrease in the orbital speed. Beyond $R = 4R_s$, the visible light from M31 was too faint for Rubin and Ford to measure the redshift; as they wrote in their original paper, "extrapolation beyond that distance is a matter of taste." At $R > 4R_s$, a small amount of atomic hydrogen is still in the disk of M31, which can be detected by means of its emission line at $\lambda = 21\,\text{cm}$. By measuring the redshift of this emission line, M. Roberts and R. Whitehurst found that the orbital speed stayed at a nearly constant value of $v(R) \approx 230\,\text{km}\,\text{s}^{-1}$ out to $R \approx 30\,\text{kpc} \approx 5R_s$, as shown by the solid dots in Figure 8.4. Since the orbital speed of the stars and gas at large radii ($R > 3R_s$) is greater than it would be if stars and gas were the only matter present, we deduce the presence of a *dark halo* within which the visible stellar disk is embedded. The

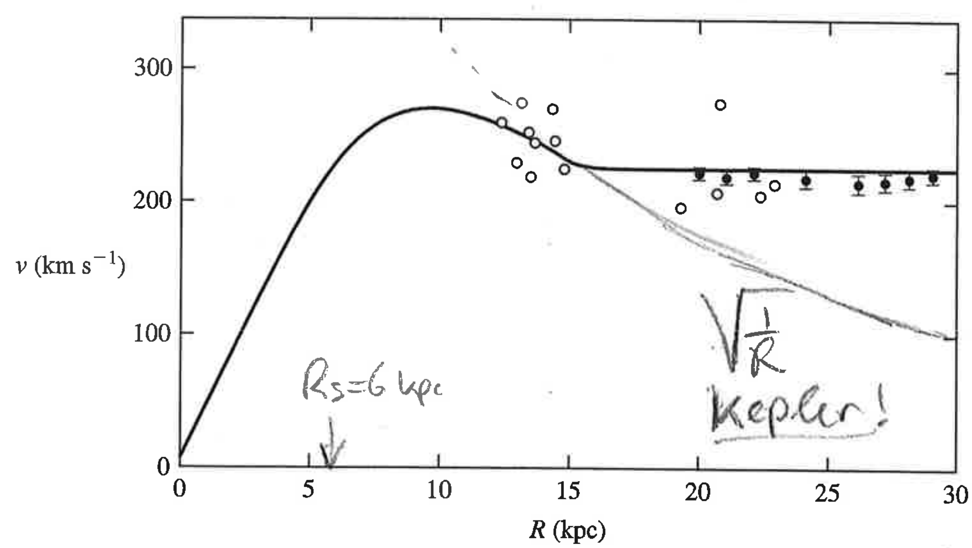


FIGURE 8.4 The orbital speed v as a function of radius in M31. The open circles show the results of Rubin and Ford (1970, ApJ, 159, 379) at visible wavelengths; the solid dots with error bars show the results of Roberts and Whitehurst (1975, ApJ, 201, 327) at radio wavelengths.

gas at large radii ($R > 3R_s$) is greater than it would be if stars and gas were the only matter present, we deduce the presence of a *dark halo* within which the visible stellar disk is embedded. The mass of the dark halo provides the necessary gravitational "anchor" to keep the high-speed stars and gas from being flung out into intergalactic space.

M31 is not a freak; most, if not all, spiral galaxies have comparable dark halos. For instance, our own galaxy has an orbital speed that actually seems to be roughly constant at $R > 15$ kpc, instead of decreasing in a Keplerian fashion. If we approximate the orbital speed v as being constant with radius, the mass of a spiral galaxy, including both the luminous disk and the dark halo, can be found from Equation 7.10:

Using (7.9):

$\frac{M}{L_V} = \frac{1.05\times10^{11}\,M_\odot}{2\times10^{10}\,L_\odot}$

$$M(R) = \frac{v^2 R}{G} = 1.05 \times 10^{11}\,\mathrm{M}_\odot \left(\frac{v}{235\,\mathrm{km\,s^{-1}}}\right)^2 \left(\frac{R}{8.2\,\mathrm{kpc}}\right). \tag{7.12}$$

$\Rightarrow \left(\frac{M}{L}\right)_{gal} \simeq 5.3$ for $R < R_{sun} \simeq 8.2$ kpc

The values of v and R in the above equation are scaled to the Sun's location in our galaxy. Since our galaxy's luminosity in the V band is estimated to be $L_{\mathrm{gal},V} = 2.0 \times 10^{10}\,\mathrm{L}_{\odot,V}$, this means that the mass-to-light ratio of our galaxy, taken as a whole, is

$5.3\times12 = 5.3\times\left(\frac{100}{8.2}\right)$

BUT $\left(\frac{M}{L}\right) \simeq 64$ at $R_h = 100$ kpc

$$\langle M/L_V \rangle_{\mathrm{gal}} \approx 64\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V} \left(\frac{R_{\mathrm{halo}}}{100\,\mathrm{kpc}}\right), \tag{7.13}$$

(8.3×12)

using $v = 235\,\mathrm{km\,s^{-1}}$ in Equation 7.12. The quantity R_{halo} is the radius of the dark halo surrounding the luminous disk of our galaxy. The exact value of R_{halo} is poorly known. A rough estimate of the halo size can be made by looking at the velocities of the globular clusters and satellite galaxies (such as the Magellanic Clouds) that orbit our galaxy. For these hangers-on to remain gravitationally bound to our galaxy, the halo must extend as far as $R_{\mathrm{halo}} \approx 75$ kpc, implying a total mass for our galaxy of $M_{\mathrm{gal}} \approx 9.6 \times 10^{11}\,\mathrm{M}_\odot$, and a total mass-to-light ratio $\langle M/L_V \rangle_{\mathrm{gal}} \approx 48\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$. This mass-to-light ratio is an order of magnitude greater than that of the stars in our galaxy, implying a dark halo much more massive than the stellar disk. Some astronomers have speculated that the dark halo is actually four times larger in radius, with $R_{\mathrm{halo}} \approx 300$ kpc; this would mean that our halo stretches nearly halfway to M31. With $R_{\mathrm{halo}} \approx 300$ kpc, the mass of our galaxy would be $M_{\mathrm{gal}} \approx 3.8 \times 10^{12}\,\mathrm{M}_\odot$, and the total mass-to-light ratio would be $\langle M/L_V \rangle_{\mathrm{gal}} \approx 190\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,V}$.

M/L of Galaxy

$R<R_c$	$M/L(<R_c)$
8.2 kpc	~5.3
75 kpc	~48
100 kpc	~64
300 kpc	~190

⇓

∃ DM halos on 100 kpc scales

7.3 Dark Matter in Clusters

The first astronomer to make a compelling case for the existence of large quantities of dark matter was Fritz Zwicky, in the 1930s. In studying the Coma cluster of galaxies (shown in Figure 7.2), he noted that the dispersion in the radial velocity of the cluster's galaxies was very large – around 1000 km s^{-1}. The stars and gas visible within the galaxies simply did not provide enough gravitational attraction to hold the cluster together. In order to keep the galaxies in the Coma cluster from flying off into the surrounding voids, Zwicky concluded, the cluster must contain a large amount of "dunkle Materie," or (translated into English) "dark matter."[5]

To follow Zwicky's reasoning at a more mathematical level, let us suppose that a cluster of galaxies consists of N galaxies, each of which can be approximated as a point mass, with a mass m_i ($i = 1, 2, \ldots, N$), a position $\vec{x}_i$, and a velocity $\dot{\vec{x}}_i$. Clusters of galaxies are gravitationally bound objects, not expanding with the Hubble flow. The motion of individual galaxies within the cluster is well described by Newtonian physics; the acceleration of the ith galaxy is thus given by the formula VECTORIZE

$$a = \sum_{\forall i \neq j} (7.8) = \sum_{j \neq i} \frac{GM(R)}{R^2} \Rightarrow \quad \ddot{\vec{x}}_i = G \sum_{j \neq i} m_j \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|^3}. \tag{7.14}$$

Note that Equation 7.14 assumes that the cluster is an isolated system, with the gravitational acceleration due to matter outside the cluster being negligibly small.

The gravitational *potential energy* of the system of N galaxies is $W = -\frac{1}{2} m v^2 = -K$

$$-\frac{1}{2}\sum_{\forall i} m_i (7.10)^2 = -\frac{1}{2}\sum m_i \frac{GM(R)}{R} \Rightarrow \quad W = -\frac{G}{2} \sum_{\substack{i,j \\ j \neq i}} \frac{m_i m_j}{|\vec{x}_j - \vec{x}_i|}. \tag{7.15}$$

(No need to vectorize W)

(If $W + K = 0$ – Not entirely correct, will lose the factor $\frac{1}{2}$ later)

This is the energy that would be required to pull the N galaxies away from each other so that they would all be at infinite distance from each other. (The factor of 1/2 in front of the double summation ensures that each pair of galaxies is only counted once in computing the potential energy.) The potential energy of the cluster can also be written in the form

$$(7.15) \Rightarrow \quad \boxed{W = -\alpha \frac{GM^2}{r_h}}, \tag{7.16}$$

where $M = \sum m_i$ is the total mass of all the galaxies in the cluster, α is a numerical factor of order unity that depends on the density profile of the cluster, and r_h is the *half-mass* radius of the cluster – that is, the radius of a sphere centered on the cluster's center of mass and containing a mass $M/2$. For observed clusters of galaxies, it is found that $\alpha \approx 0.45$ gives a good fit to the potential energy. $(\alpha \simeq \frac{1}{2})$

The *kinetic energy* associated with the relative motion of the galaxies in the cluster is

$$K = \frac{1}{2} \sum_i m_i |\dot{\vec{x}}_i|^2 = \frac{1}{2}\sum_i m_i v_i^2 \tag{7.17}$$

[5] Although Zwicky's work popularized the phrase "dark matter," he was not the first to use it in an astronomical context. For instance, in 1908, Henri Poincaré discussed the possible existence within our galaxy of "matière obscure" (rendered as "dark matter" in the standard English translation of Poincaré's works).

The kinetic energy K can also be written in the form

$$\boxed{K = \frac{1}{2}M\langle v^2\rangle,} \tag{7.18}$$

where

Mass-weighted average v^2 or mean square velocity:

$$\langle v^2\rangle \equiv \frac{1}{M}\sum_i m_i|\dot{\vec{x}}_i|^2 \tag{7.19}$$

is the mean square velocity (weighted by galaxy mass) of all the galaxies in the cluster.

It is also useful to define the *moment of inertia* of the cluster as

$\dot{I} = \sum_i m_i 2|\vec{x}_i|\,\dot{\vec{x}}_i$

$\ddot{I} = 2\sum_i m_i (|\dot{\vec{x}}_i|\,\dot{\vec{x}}_i + |\vec{x}_i|\,\ddot{\vec{x}}_i)$

$$\boxed{I \equiv \sum_i m_i|\vec{x}_i|^2.} \tag{7.20}$$

Answers the Question: "Where exactly is the cluster mass w.r.t. the center?"

The moment of inertia I can be linked to the kinetic energy and the potential energy if we start by taking the second time derivative of I:

$$\ddot{I} = 2\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i + \dot{\vec{x}}_i \cdot \dot{\vec{x}}_i). \tag{7.21}$$

(7.17) $\left[K = \frac{1}{2}\sum_i m_i v_i^2\right]$

Using Equation 7.17, we can rewrite this as

$$\ddot{I} = 2\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) + 4K = 2W + 4K \tag{7.22}$$

To introduce the potential energy W into the above relation, we can use Equation 7.14 to write

$$\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) = G\sum_{\substack{i,j\\ j\neq i}} m_i m_j \frac{\vec{x}_i \cdot (\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3}. \tag{7.23}$$

However, we could equally well switch around the i and j subscripts to find the equally valid equation

$$\sum_j m_j(\vec{x}_j \cdot \ddot{\vec{x}}_j) = G\sum_{\substack{j,i\\ i\neq j}} m_j m_i \frac{\vec{x}_j \cdot (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} = -G\sum \frac{m_i m_j (\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3} \tag{7.24}$$

with i,j swapped!

[ie, force $|\vec{x}_j - \vec{x}_i|$ in denominator]

Since

$$\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) = \sum_j m_j(\vec{x}_j \cdot \ddot{\vec{x}}_j) \tag{7.25}$$

(7.23) + (7.24)

(it doesn't matter whether we call the variable over which we're summing i or j or k or "Fred"), we can combine Equations 7.23 and 7.24 to find

i.e., don't use (7.24) or (7.25) but use "half" of each! :)

$$\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) = \frac{1}{2}\left[\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) + \sum_j m_j(\vec{x}_j \cdot \ddot{\vec{x}}_j)\right]$$

$\frac{G}{2}\sum_{i,j} m_i m_j \left[\frac{-x_i^2 + 2x_i x_j - x_j^2}{|x_i - x_j|^3}\right]$

$$= -\frac{G}{2}\sum_{i,j} m_i m_j \frac{(x_i - x_j)^2}{|x_i - x_j|^3} = -\frac{G}{2}\sum_{i,j} m_i m_j \frac{(\vec{x}_j - \vec{x}_i)^2}{|\vec{x}_j - \vec{x}_i|^3} \Rightarrow -\frac{G}{2}\sum_{i,j} \frac{m_i m_j}{|\vec{x}_j - \vec{x}_i|} = W. \tag{7.26}$$

Thus, the first term on the right-hand side of Equation 7.22 is simply $2W$, and we may now write down the simple relation

$$(7.22) \Longrightarrow \qquad \boxed{\ddot{I} = 2W + 4K.} \qquad (7.27)$$

This relation is known as the *virial theorem*. It was first derived in the nineteenth century in the context of the kinetic theory of gases, but as we have seen, it applies perfectly well to a self-gravitating system of point masses.

The virial theorem is particularly useful when it is applied to a system in steady state, with a constant moment of inertia. (This implies, among other things, that the system is neither expanding nor contracting, and that we are using a coordinate system in which the center of mass of the cluster is at rest.) If $I =$ constant, then the *steady-state virial theorem* is

$$\ddot{I} = \dot{I} = 0 \Longrightarrow \qquad \frac{0}{2} = W + 2K, \qquad (7.28)$$

or

The correct result $\qquad K = -\frac{W}{2}. \qquad (7.29)$ (we incorrectly assumed $W = -K$ in (7.15))

Using Equations 7.16 and 7.18 in Equation 7.29, we find

$$\boxed{\frac{1}{2}M\langle v^2\rangle = \frac{\alpha}{2}\frac{GM^2}{r_h}.} \qquad (7.30)$$

This means we can use the virial theorem to estimate the mass of a cluster of galaxies, or any other self-gravitating steady-state system:

$$(7.12) \Longleftrightarrow \qquad \boxed{M = \frac{\langle v^2\rangle r_h}{\alpha G}.} \Rightarrow \langle v\rangle = \sqrt{\frac{\alpha G M}{r_h}} \qquad (7.31)$$

where $\alpha(\simeq 1)$ depends on how non-spherical the cluster is. On average $\alpha \simeq 0.5$

Note the similarity between Equation 7.12, used to estimate the mass of a rotating spiral galaxy, and Equation 7.31, used to estimate the mass of a cluster of galaxies. In either case, we estimate the mass of a self-gravitating system by multiplying the square of a characteristic velocity by a characteristic radius, then dividing by the gravitational constant G.

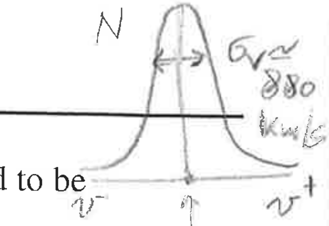
Applying the virial theorem to a real cluster of galaxies, such as the Coma cluster, is complicated by the fact that we have only partial information about the cluster, and thus do not know $\langle v^2\rangle$ and r_h exactly. For instance, we can find the line-of-sight velocity of each galaxy from its redshift, but the velocity perpendicular to the line of sight is unknown. From measurements of the redshifts of hundreds of galaxies in the Coma cluster, the mean redshift of the cluster is found to be

$$\langle z\rangle = 0.0232, \Rightarrow \langle v_r\rangle = c\langle z\rangle = 6960 \text{ km s}^{-1} \qquad (7.32)$$

which can be translated into a distance

$$d_{\text{Coma}} = (c/H_0)\langle z\rangle = 102\,\text{Mpc}. \qquad (7.33)$$

(for $H_0 = 68$ km/sec/Mpc)!



The velocity dispersion of the cluster along the line of sight is found to be

$$\sigma_r = \langle (v_r - \langle v_r \rangle)^2 \rangle^{1/2} = 880\,\mathrm{km\,s^{-1}}. \tag{7.34}$$

$v_{Coma} = cz_c = 6960$ km/s

If we assume that the velocity dispersion is isotropic, then the three-dimensional mean square velocity $\langle v^2 \rangle$ will be equal to three times the one-dimensional mean square velocity σ_r^2, yielding

$$\langle v^2 \rangle = 3(880\,\mathrm{km\,s^{-1}})^2 = 2.32 \times 10^{12}\,\mathrm{m^2\,s^{-2}}. \tag{7.35}$$

Estimating the half-mass radius r_h of the Coma cluster is even more peril-ridden than estimating the mean square velocity $\langle v^2 \rangle$. After all, we don't know the distribution of dark matter in the cluster beforehand; in fact, the total amount of dark matter is what we're trying to find out. However, if we assume that the mass-to-light ratio is constant with radius, then the sphere containing half the mass of the cluster will be the same as the sphere containing half the luminosity of the cluster. If we further assume that the cluster is intrinsically spherical, then the observed distribution of galaxies within the Coma cluster indicates a half-mass radius

$$r_h \approx 1.5\,\mathrm{Mpc} \approx 4.6 \times 10^{22}\,\mathrm{m}. = \frac{\text{Abell radius}}{2} \tag{7.36}$$

$= \frac{3\,\mathrm{Mpc}}{2}$

After all these assumptions and approximations, we may estimate the mass of the Coma cluster to be

(7.31) ⇒ pg 127

$$M_{\mathrm{Coma}} = \frac{\langle v^2 \rangle r_h}{\alpha G} \approx \frac{(2.32 \times 10^{12}\,\mathrm{m^2\,s^{-2}})(4.6 \times 10^{22}\,\mathrm{m})}{(0.45)(6.67 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}})}$$
$$\approx 4 \times 10^{45}\,\mathrm{kg} \approx 2 \times 10^{15}\,\mathrm{M_\odot} = M_{Coma}^{TOTAL} \tag{7.37}$$

$\left(\frac{M_*}{M_{TOT}}\right)_{Coma} \simeq \frac{2\times10^{13}}{2\times10^{15}} = \frac{1}{100}$

Thus, about one percent of the mass of the Coma cluster consists of stars ($M_{\mathrm{Coma},\star} \approx 2 \times 10^{13}\,\mathrm{M_\odot}$), and about ten percent consists of hot intracluster gas ($M_{\mathrm{Coma,gas}} \approx 2 \times 10^{14}\,\mathrm{M_\odot}$). Combined with the luminosity of the Coma cluster, $L_{\mathrm{Coma},V} \approx 5 \times 10^{12}\,\mathrm{L_{\odot,V}}$, the total mass of the Coma cluster implies a mass-to-light ratio

$$\langle M/L_V \rangle_{\mathrm{Coma}} \sim 400\,\mathrm{M_\odot}/\,\mathrm{L_{\odot,V}}, \simeq \frac{2\times10^{15}M_\odot}{5\times10^{12}L_{\odot,V}} \tag{7.38}$$

greater than the mass-to-light ratio of our galaxy.

The presence of a vast reservoir of dark matter in the Coma cluster is confirmed by the fact that the hot, X-ray emitting intracluster gas, shown in Figure 7.3, is still in place; if there were no dark matter to anchor the gas gravitationally, the hot gas would have expanded beyond the cluster on time scales much shorter than the Hubble time. The temperature and density of the hot gas in the Coma cluster can be used to make yet another estimate of the cluster's mass. If the hot intracluster gas is supported by its own pressure against gravitational infall, it must obey the equation of hydrostatic equilibrium:

(4.55) & footnote 7 (p.60) ⇓

$P_{gas} = \frac{GM(r)\rho_{gas}}{r} = \varepsilon_{HOT}$ ⇒

$$\frac{dP_{\mathrm{gas}}}{dr} = -\frac{GM(r)\rho_{\mathrm{gas}}(r)}{r^2} \tag{7.39}$$

(atomic weight)

where P_{gas} is the pressure of the gas, ρ_{gas} is the density of the gas, and M is the *total* mass inside a sphere of radius r, including gas, stars, dark matter, lost socks, and anything else.

The pressure of the gas is given by the perfect gas law,

$$\frac{dP}{dr} = \frac{-GM(r).P\mu m_p}{r^2.kT(r)} \; (7.39) \Leftarrow P_{gas} = \frac{P\mu m_p}{kT} \Leftarrow \quad P_{\mathrm{gas}} = \frac{\rho_{\mathrm{gas}} k T_{\mathrm{gas}}}{\mu} \Leftarrow PV = NkT \text{ with } \rho_{gas} = \frac{N.\mu}{V} = \frac{M}{V} \tag{7.40}$$

where T_{gas} is the temperature of the gas, and μ is the mean mass per gas particle. The mass of the cluster, as a function of radius, is found by combining Equations 7.39 and 7.40:

$$M(r) = \frac{kT(r).r^2}{G\mu m_p}\cdot\boxed{\frac{1}{P}\frac{dP}{dr}} \Rightarrow \quad M(r) = \frac{kT_{\mathrm{gas}}(r) r}{G\mu}\left[-\frac{d\ln\rho_{\mathrm{gas}}}{d\ln r} - \frac{d\ln T_{\mathrm{gas}}}{d\ln r}\right]. \tag{7.41}$$

$P = \frac{\rho kT}{\mu m_p}$

The above equation assumes that μ is constant with radius, as we'd expect if the chemical composition and ionization state of the gas are uniform throughout the cluster.

The X-rays emitted from the hot intracluster gas are a combination of bremsstrahlung emission (caused by the acceleration of free electrons by protons and helium nuclei) and line emission from highly ionized iron and other heavy elements. Starting from an X-ray spectrum, it is possible to fit models to the emission and thus compute the temperature $T_{\mathrm{gas}}(r)$, density $\rho_{\mathrm{gas}}(r)$, and chemical composition of the gas. Using this technique, the mass of the Coma cluster is estimated to be $M \approx 1.3 \times 10^{15}\,\mathrm{M}_\odot$ within $r \approx 4\,\mathrm{Mpc}$ of the cluster center. Given the uncertainties, this is consistent with the mass estimate of the virial theorem. $T^{Coma}_{gas} \simeq 10^{7.5} - 10^{8}\,K$

Other clusters of galaxies besides the Coma cluster have had their masses estimated, using the virial theorem applied to their galaxies or the equation of hydrostatic equilibrium applied to their gas. Typical mass-to-light ratios for rich clusters are similar to those of the Coma cluster. If the masses of all the clusters of galaxies are added together, it is found that their density parameter is

$$\boxed{\Omega_{\mathrm{clus},0} \approx 0.2.} \quad \Omega_m \text{ (Planck 2016)} = 0.32 \tag{7.42}$$

This provides a *lower limit* to the matter density of the universe, since any smoothly distributed matter in the intercluster voids will not be included in this number.

7.4 Gravitational Lensing

So far, I have outlined the classical methods for detecting dark matter via its gravitational effects on luminous matter.[6] We can detect dark matter around spiral

[6] The roots of these methods can be traced back as far as the year 1846, when Leverrier and Adams deduced the existence of the dim planet Neptune by its effect on the orbit of Uranus.

galaxies because it affects the motions of stars and interstellar gas. We can detect dark matter in clusters of galaxies because it affects the motions of galaxies and intracluster gas. However, as Einstein realized, dark matter will affect not only the trajectory of matter, but also the trajectory of photons. Thus, dark matter can bend and focus light, acting as a *gravitational lens*. The effects of dark matter on photons have been used to search for dark matter within the halo of our own galaxy, as well as in distant clusters of galaxies.

To see how gravitational lensing can be used to detect dark matter, start by considering the dark halo surrounding our galaxy. If there were a population of cold white dwarfs, black holes, brown dwarfs, or similar dim compact objects in the halo, they would be very difficult to detect from the light that they emit. Thus, it was suggested that part of the dark matter in the halo could consist of MACHOs, a slightly strained acronym for MAssive Compact Halo Objects. If a photon passes such a compact massive object at an impact parameter b, as shown in Figure 7.4, the local curvature of spacetime will cause the photon to be deflected by an angle (only shows that 7.43 is dimensionally correct)

(7.29) $2K = -W \Rightarrow 2 \times \frac{1}{2} mc^2 = \frac{GmM}{b} \Rightarrow$

$$\alpha = \frac{4GM}{c^2 b}, \simeq \frac{4v^2}{c^2} \text{ with } v = \sqrt{\frac{GM}{b}} \leftarrow \text{Kepler} \qquad (7.43)$$

where M is the mass of the compact object. For instance, light from a distant star that just grazes the Sun's surface should be deflected through an angle

$$\boxed{\alpha \simeq \frac{4GM}{c^2 d}} \Leftarrow R_\odot = b \simeq \alpha . d \quad \text{Fig 7:4}$$

$$\boxed{\alpha = \frac{4GM_\odot}{c^2 R_\odot} = 1.7\,\text{arcsec.}} \qquad (7.44)$$

In 1919, after Einstein predicted a deflection of this magnitude, an eclipse expedition photographed stars in the vicinity of the Sun. Comparison of the eclipse photographs with photographs of the same star field taken six months earlier revealed that the apparent positions of the stars were deflected by the amount that Einstein had predicted. This result brought fame to Einstein and experimental support to the theory of general relativity.

Used in: Longair (4.34) Ryden (7.45)

Since a massive object can deflect light, it can act as a lens. Suppose a MACHO in the halo of our galaxy passes directly between an observer in our galaxy and a star in the Large Magellanic Cloud. Figure 7.5 shows such a situation, with a MACHO halfway between the observer and the star. As the MACHO deflects the light from the distant star, it produces an image of the star that is both distorted and amplified. If the MACHO is *exactly* along the line of sight between the observer and the lensed star, the image produced is a perfect Einstein ring, with angular radius

⊛ Per unit time:
* Photon moves length c/sec to the right
* and length v/sec down due to $\frac{GM}{b} \Rightarrow$
$\Rightarrow \alpha^2 \simeq GM/b$

in 1sec: $d = c$
$\Rightarrow \alpha \simeq \frac{b}{d} \Rightarrow$
$b \simeq \alpha d$

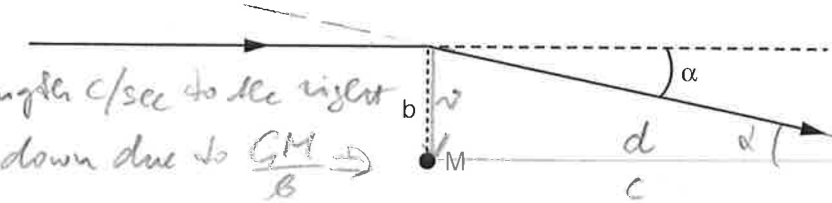


Figure 7.4 Deflection of light by a massive compact object.

$$\Delta t = \frac{d\,\theta_E}{2v} \approx 90\,\text{days}\left(\frac{M}{1\,\text{M}_\odot}\right)^{1/2}\left(\frac{v}{200\,\text{km s}^{-1}}\right)^{-1}, \tag{7.47}$$

where v is the relative transverse velocity of the MACHO and the lensed star as seen by the observer on Earth. Generally speaking, more massive MACHOs produce larger Einstein rings and thus will amplify the lensed star for a longer time.

The research groups that searched for MACHOs found a scarcity of short duration lensing events, suggesting that there is no significant population of brown dwarfs or freefloating planets (with $M < 0.08\,\text{M}_\odot$) in the dark halo of our galaxy. The total number of lensing events they found suggests that at most 8 percent of the halo mass could be in the form of MACHOs. The general conclusion is that most of the matter in the dark halo of our galaxy is due to a smooth distribution of nonbaryonic dark matter, instead of being congealed into MACHOs of roughly stellar or planetary mass.

Gravitational lensing occurs at all mass scales. Suppose, for instance, that a cluster of galaxies, with $M \sim 10^{14}\,\text{M}_\odot$, at a distance $\sim 500\,\text{Mpc}$ from our galaxy, lenses a background galaxy at $d \sim 1000\,\text{Mpc}$. The Einstein radius for this configuration will be

$d_A(z_{gxy} \simeq 0.2)$ $d_A(z_{cl} \simeq 0.14)$

30''

Einstein Angle (7.45, 7.46)

$$\theta_E \approx 0.5\,\text{arcmin}\left(\frac{M}{10^{14}\,\text{M}_\odot}\right)^{1/2}\left(\frac{d}{1000\,\text{Mpc}}\right)^{-1/2}. \tag{7.48}$$

The arc-shaped images into which the background galaxy is distorted by the lensing cluster can thus be resolved. For instance, Figure 7.6 shows a *Hubble*

Figure 7.6 The central regions of the rich cluster Abell 2218, displaying gravitationally lensed arcs. The region shown is 3.2 arcminutes by 1.6 arcminutes, equivalent to 0.68 Mpc by 0.34 Mpc at the distance of Abell 2218. [NASA, ESA, and Johan Richard (Caltech)]

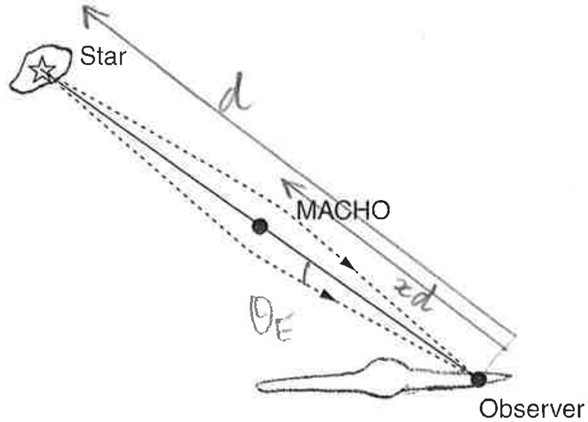


Figure 7.5 Light from a star in the Large Magellanic Cloud is deflected by a MACHO on its way to an observer in the disk of our galaxy (seen edge-on in this figure).

For $x \simeq 0.5$ ⟺

$$\theta_E = \left(\frac{4GM}{c^2 d} \frac{1-x}{x} \right)^{1/2}, \tag{7.45}$$

where M is the mass of the lensing MACHO, d is the distance from the observer to the lensed star, and xd (where $0 < x < 1$) is the distance from the observer to the lensing MACHO. The angle θ_E is known as the *Einstein radius*. If $x \approx 0.5$ (that is, if the MACHO is roughly halfway between the observer and the lensed star), then

0.0004

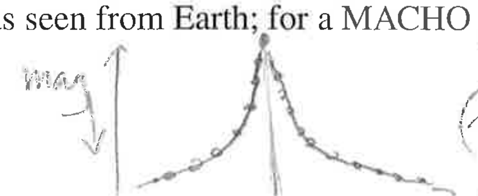
$$\theta_E \approx 4 \times 10^{-4} \text{ arcsec} \left(\frac{M}{1\,\mathrm{M}_\odot} \right)^{1/2} \left(\frac{d}{50\,\mathrm{kpc}} \right)^{-1/2}. \tag{7.46}$$

If the MACHO does not lie perfectly along the line of sight to the star, then the image of the star will be distorted into two or more arcs instead of a single unbroken ring. Although the Einstein radius for an LMC star being lensed by a MACHO is too small to be resolved, it is possible, in some cases, to detect the amplification of the flux from the star. For the amplification to be significant, the angular distance between the MACHO and the lensed star, as seen from Earth, must be comparable to, or smaller than, the Einstein radius. Given the small size of the Einstein radius, the probability of any particular star in the LMC being lensed at any moment is tiny. It has been calculated that if the dark halo of our galaxy were entirely composed of MACHOs, then the probability of any given star in the LMC being lensed at any given time would still only be $P \sim 5 \times 10^{-7}$.

To detect lensing by MACHOs, various research groups took up the daunting task of monitoring millions of stars in the Large Magellanic Cloud to watch for changes in their flux. Since the MACHOs in our dark halo and the stars in the LMC are in constant relative motion, the typical signature of a "lensing event" is a star that becomes brighter as the angular distance between star and MACHO decreases, then becomes dimmer as the angular distance increases again. The typical time scale for a lensing event is the time it takes a MACHO to travel through an angular distance equal to θ_E as seen from Earth; for a MACHO halfway between here and the LMC, this is

caustic transit magnification

$\mu = \frac{\text{const}}{\sqrt{d}}$ (see W18)



sured which are independent of their redshifts, enabling direct estimates of Hubble's constant to be made (see Sect. 8.3).

From Longair's Book

4.3.4 Gravitational Lensing by Clusters of Galaxies

(More exact Alan Ryden)

A beautiful method for determining the mass distribution in clusters of galaxies has been provided by the discovery of gravitationally lensed arcs about the central regions of rich clusters of galaxies. In his great paper of 1915 on the General Theory of Relativity, Einstein showed that the deflection of light by the Sun amounts to precisely twice that predicted by a simple Newtonian calculation. According the General Relativity, the angular deflection of the position of a background star due to the bending of space-time by a point mass M is

$$\alpha = \frac{4GM}{pc^2}, \tag{4.30}$$

where p is the 'collision parameter' (Fig. 4.8a). For the very small deflections involved in the gravitational lens effect, p is almost exactly the distance of closest approach of the light rays to the deflector.

Chwolson (1924) and Einstein (1936) realised that, if the background star were precisely aligned with the deflecting point object, the gravitational deflection of the light rays would result in a circular ring, centred upon the deflector (Fig. 4.8b). It is a straightforward calculation to work out the radius of what came to be known as an 'Einstein ring', although they should perhaps be known as 'Chwolson rings'. In Fig. 4.8b, the angular size distance of the background source is D_s and that of the deflector, or lens, D_d, the distance between them being D_{ds}. Suppose the observed angular radius of the Einstein ring is θ_E as illustrated in Fig. 4.8b. Then, by simple geometry, since all the angles are small,

D = angular size distance! using (6.37) & Fig 6.4

$$\theta_E = \alpha\left(\frac{D_{ds}}{D_s}\right), \tag{4.31}$$

where α is the deflection given by (4.30). Therefore,

Longair Fig 4.8b $\longrightarrow \theta_E = \frac{p}{D_d}$

$$\theta_E = \alpha\left(\frac{D_{ds}}{D_s}\right) = \frac{4GM}{pc^2}\left(\frac{D_{ds}}{D_s}\right). \tag{4.32}$$

Since $p = \theta_E D_d$,

$$\theta_E^2 = \frac{4GM}{c^2}\left(\frac{D_{ds}}{D_s D_d}\right) = \frac{4GM}{c^2}\frac{1}{D}, \tag{4.33}$$

"Fat D"

where $D = (D_s D_d / D_{ds})$. Thus, the *Einstein angle* θ_E, the angle subtended by the Einstein ring at the observer, is given by

$$\theta_E = \left(\frac{4GM}{c^2}\right)^{1/2}\frac{1}{D^{1/2}}. \tag{4.34}$$

θ_E = Einstein angle $\propto \sqrt{\frac{M}{D}}$ is maximally large when

1) M is as large as possible $\Longleftrightarrow$ rich cluster with $M \gtrsim 10^{15} M_\odot$

2) D is minimized $\Longleftrightarrow$ minimize $D_s, D_d \Rightarrow$ z_s = large, $z_s \gtrsim 7$; $z_d = z_d$ = modest $z_d \lesssim 0.3-0.5$

maximize $D_{ds} \Rightarrow$ [illegible]

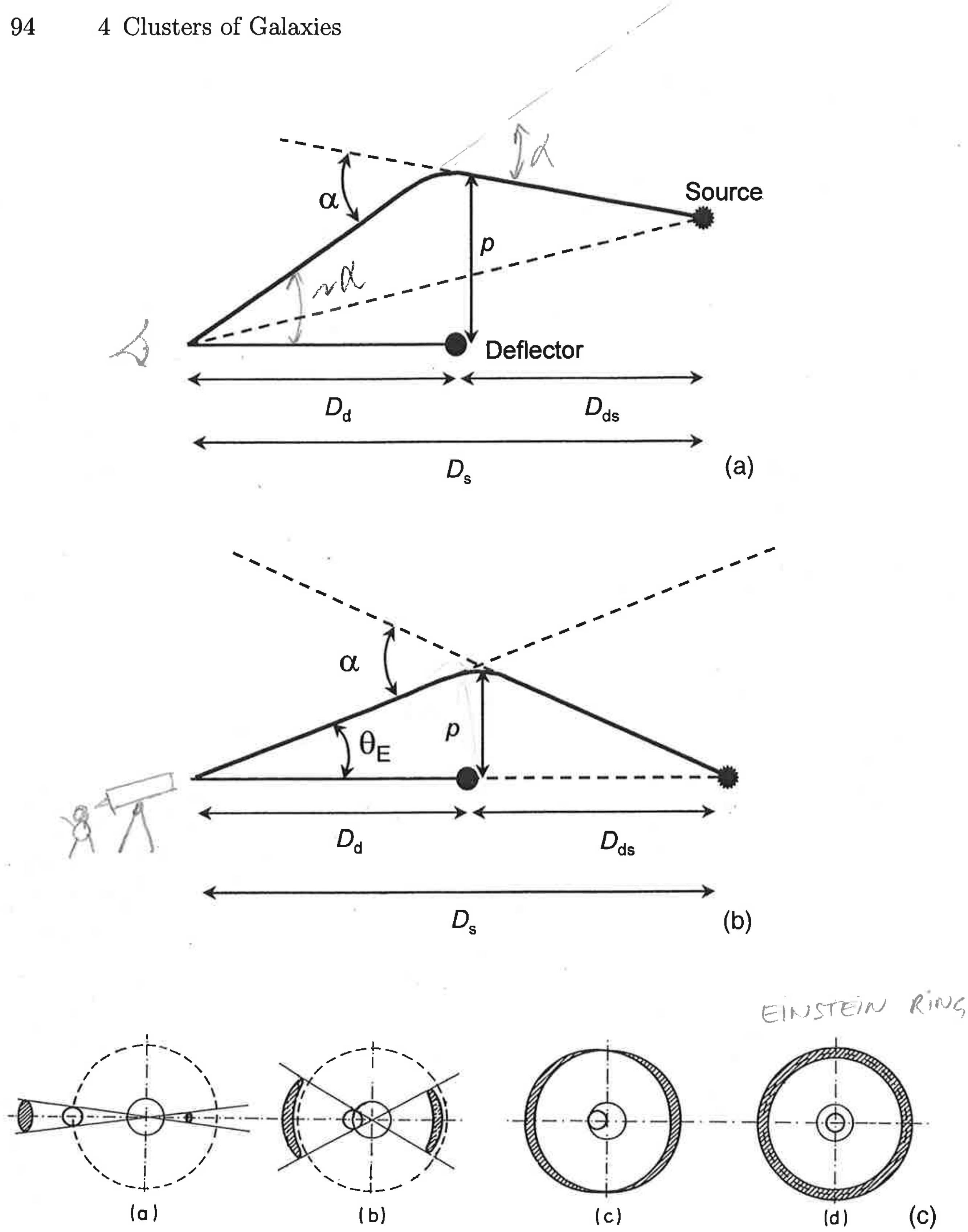


Fig. 4.8. (**a**) Illustrating the geometry of the deflection of light by a deflector, or lens, of mass M. (**b**) Illustrating the formation of an Einstein ring when the source and deflector are perfectly aligned. (**c**) Illustrating the changes of the appearance of a compact background source as it passes behind a point mass. The dashed circles correspond to the Einstein radius. When the lens and the background source are precisely aligned, an Einstein ring is formed with radius equal to the Einstein radius

Fig. 4.10. The gravitational distortions of a background source (Panel I) when it is located at different positions with respect to the axis of the gravitational lens. In this example, the lens is an ellipsoidal non-singular squeezed isothermal sphere. The ten positions of the source with respect to the critical inner and outer caustics are shown in the panel (S). The panels labelled (1) to (10) show the shapes of the images of the lensed source (from J.-P. Kneib, Ph.D. Thesis (1993)). Note the shapes of the images when the source crosses the critical caustics. Positions (6) and

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Space Telescope image of the cluster Abell 2218, which has a redshift $z = 0.176$, and hence is at a proper distance $d = 740\,\text{Mpc}$. The elongated, slightly curved arcs seen in Figure 7.6 are not oddly shaped galaxies within the cluster; instead, they are background galaxies, at redshifts $z > 0.176$, which are gravitationally lensed by the cluster mass. The mass of clusters can be estimated by the degree to which they lens background galaxies. The masses calculated in this way are in general agreement with the masses found by applying the virial theorem to the motions of galaxies in the cluster or by applying the equation of hydrostatic equilibrium to the hot intracluster gas.

Chapter 8 + Planck 2016 ? $\Rightarrow$ $\Omega_{dm,0} \simeq 0.262$

$\Omega_{b,0} \simeq 0.048 \leftarrow (7.6)$, Also chapter 9

7.5 What's the Matter?

TOTAL $\Omega_{m,0} \simeq 0.310$

We described how to detect dark matter by its gravitational effects, but have been dodging the essential question: "What is it?" As you might expect, conjecture about the nature of the nonbaryonic dark matter has run rampant (some might even say it has run amok). A component of the universe that is totally invisible is an open invitation to speculation. To give a taste of the variety of speculation, some scientists have proposed that the dark matter might be made of axions, a type of elementary particle with a rest energy of $m_{\text{ax}}c^2 \sim 10^{-5}\,\text{eV}$, equivalent to $m_{\text{ax}} \sim 2 \times 10^{-41}\,\text{kg}$. This is a rather low mass – it would take some 50 billion axions (if they indeed exist) to equal the mass of one electron. On the other hand, some scientists have conjectured that the dark matter might be made of primordial black holes, with masses up to $m_{\text{BH}} \sim 10^5\,\text{M}_\odot$, equivalent to $m_{\text{BH}} \sim 2 \times 10^{35}\,\text{kg}$.[7] This is a rather high mass – it would take some 30 billion Earths to equal the mass of one primordial black hole (if they indeed exist). It is a sign of the vast ignorance concerning nonbaryonic dark matter that two candidates for the role of dark matter differ in mass by 76 orders of magnitude.

One nonbaryonic particle that we know exists, and which has a nonzero mass, is the neutrino. As stated in Section 5.1, there should exist today a cosmic background of neutrinos. Just as the cosmic microwave background is a relic of the time when the universe was opaque to photons, the cosmic neutrino background is a relic of the time when the universe was hot and dense enough to be opaque to neutrinos. The number density of each of the three flavors of neutrinos (ν_e, ν_μ, and ν_τ) has been calculated to be $3/11$ times the number density of CMB photons, yielding a total number density of neutrinos (2.35) $\leftarrow n_\gamma = n_{CMB} = \beta T^3 \Leftarrow$ Eq (2.31) + (2.32)

(6.16) $\Rightarrow$ etc

$$n_\nu = 3\left(\frac{3}{11}\right)n_\gamma = \left(\frac{9}{11}\right)(4.108 \times 10^8\,\text{m}^{-3}) = 3.36 \times 10^8\,\text{m}^{-3}. \qquad (7.49)$$

This means that at any instant, about twenty million cosmic neutrinos are zipping through your body, "like photons through a pane of glass." In order to

[7] A *primordial* black hole is one that formed very early in the history of the universe, rather than by the collapse of a massive star later on.

provide *all* the nonbaryonic mass in the universe, the average neutrino mass would have to be

(5.12)

$$m_\nu c^2 = \frac{\Omega_{\text{dm},0}\varepsilon_{c,0}}{n_\nu}. \quad (7.50)$$

Chapter 8

$\varepsilon_{c,0} = \frac{3c^2}{8\pi G} H_0^2$

Given a density parameter in nonbaryonic dark matter of $\Omega_{\text{dm},0} \approx 0.262$, this implies that an average neutrino mass of

(4.31)

$\Omega_{B,0} \simeq 0.048 \leftarrow (7.6)$

$+$

$\Omega_{m,0} \simeq 0.31 \leftarrow$ Planck 2016!

(4.31) $\varepsilon_{c,0} \simeq 4870\ \text{MeV/m}^3$

(7.49) $n_\nu \simeq 3.36 \times 10^8/\text{m}^3$

$$m_\nu c^2 \approx \frac{0.262(4870\,\text{MeV m}^{-3})}{3.36 \times 10^8\,\text{m}^{-3}} \approx 3.8\,\text{eV} \quad (7.51)$$

(7.49)

would be necessary to provide all the nonbaryonic dark matter in the universe. Studies of neutrino oscillations and of the large scale structure of the universe (see Equations 2.25 and 2.26) indicate that the average neutrino mass actually lies in the range

(2.25)/3 (2.26)/3 [/3 because only e neutrino's considered not μ and τ]

$$0.019\,\text{eV} < m_\nu c^2 < 0.1\,\text{eV}. \quad (7.52)$$

This implies that the current density parameter in massive neutrinos lies in the range

using (4.31), (4.32) ⇒

$$0.0013 < \Omega_{\nu,0} < 0.007, \quad (7.53)$$

and that less than 3 percent of the dark matter takes the form of neutrinos.

Given the insufficient mass density of neutrinos, particle physicists have provided several possible alternative candidates for the role of dark matter. For instance, consider the extension of the Standard Model of particle physics known as supersymmetry. Various supersymmetric models predict the existence of massive nonbaryonic particles such as photinos, gravitinos, axinos, sneutrinos, gluinos, and so forth. Like neutrinos, the hypothetical supersymmetric particles interact with other particles only through gravity and through the weak nuclear force, which makes them intrinsically difficult to detect. Particles that interact via the weak nuclear force, but which are much more massive than the upper limit on the neutrino mass, are known generically as Weakly Interacting Massive Particles, or WIMPs.[8] Since WIMPs, like neutrinos, do interact with atomic nuclei on occasion, experimenters have set up WIMP detectors to discover cosmic WIMPs. So far (to repeat a statement made in the first edition of this book), no convincing detections have been made – but the search goes on.

Exercises

7.1 Suppose it were suggested that black holes of mass $10^{-8}\,M_\odot$ made up all the dark matter in the halo of our galaxy. How far away would you expect the nearest such black hole to be? How frequently would you expect such a