6

Measuring Cosmological Parameters

Cosmologists would like to know the scale factor $a(t)$ for the universe. For a model universe whose contents are known with precision, the scale factor can be computed from the Friedmann equation. Finding $a(t)$ for the real universe, however, is much more difficult. The scale factor is not directly observable; it can only be deduced indirectly from the imperfect and incomplete observations that we make of the universe around us.

In the previous chapter, I pointed out that if we knew the energy density ε for each component of the universe, we could use the Friedmann equation to find the scale factor $a(t)$. The argument works in the other direction, as well; if we could determine $a(t)$ from observations, we could use that knowledge to find ε for each component. Let's see, then, what constraints we can put on the scale factor by making observations of distant astronomical objects.

6.1 "A Search for Two Numbers"

Since determining the exact functional form of $a(t)$ is difficult, it is useful, instead, to do a Taylor series expansion for $a(t)$ around the present moment. The complete Taylor series is

$$a(t) = a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0} (t - t_0) + \frac{1}{2} \left.\frac{d^2a}{dt^2}\right|_{t=t_0} (t - t_0)^2 + \cdots \qquad (6.1)$$

To exactly reproduce an arbitrary function $a(t)$ for all values of t, an infinite number of terms is required in the expansion. However, the usefulness of a Taylor series expansion resides in the fact that if a doesn't fluctuate wildly with t, using only the first few terms of the expansion gives a good approximation in the immediate vicinity of t_0. The scale factor $a(t)$ is a good candidate for a Taylor expansion. The different model universes examined in the previous two chapters all had smoothly varying scale factors, and there's no evidence that the real universe has a wildly oscillating scale factor.

Keeping the first three terms of the Taylor expansion, the scale factor in the recent past and the near future can be approximated as

$$a(t) \approx a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0} (t - t_0) + \frac{1}{2} \left.\frac{d^2a}{dt^2}\right|_{t=t_0} (t - t_0)^2. \tag{6.2}$$

Dividing by the current scale factor, $a(t_0)$,

$$\frac{a(t)}{a(t_0)} \approx 1 + \left.\frac{\dot{a}}{a}\right|_{t=t_0} (t - t_0) + \frac{1}{2} \left.\frac{\ddot{a}}{a}\right|_{t=t_0} (t - t_0)^2. \tag{6.3}$$

Using the normalization $a(t_0) = 1$, this expansion for the scale factor is customarily written in the form

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2. \tag{6.4}$$

In Equation 6.4, the parameter H_0 is our old acquaintance the Hubble constant,

(6.4) ⇒

$$H_0 \equiv \left.\frac{\dot{a}}{a}\right|_{t=t_0}, \tag{6.5}$$

and the parameter q_0 is a dimensionless number called the *deceleration parameter*, defined as

$$q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_{t=t_0} = -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}. \tag{6.6}$$

Decelerate!

Note the choice of sign in defining q_0. A positive value of q_0 corresponds to $\ddot{a} < 0$, meaning that the universe's expansion is decelerating (that is, the relative velocity of any two points is decreasing). A negative value of q_0 corresponds to $\ddot{a} > 0$, Accelerate! meaning that the relative velocity of any two points is increasing with time. The choice of sign for q_0, and the fact that it's named the *deceleration* parameter, is because it was first defined during the mid-1950s, when the limited information available favored a matter-dominated universe with $\ddot{a} < 0$. If the universe contains a sufficiently large cosmological constant, however, the deceleration parameter q_0 can have either sign.

The Taylor expansion of Equation 6.4 is physics-free. It is simply a mathematical description of how the universe expands at times $t \sim t_0$, and says nothing at all about what forces act to accelerate the expansion (to take a Newtonian viewpoint of the physics involved). In a famous 1970 review article, the observational cosmologist Allan Sandage described all of cosmology as "a search for two numbers." Those two numbers were H_0 and q_0. Although the scope of cosmology has widened considerably since Sandage wrote his article, it is still possible to describe the recent expansion of the universe in terms of H_0 and q_0.

Although H_0 and q_0 are themselves free of the theoretical assumptions underlying the Friedmann and acceleration equations, we can use the acceleration equation to predict what q_0 will be in a given model universe. If our model universe contains N components, each with a different value of the equation-of-state parameter w_i, the acceleration equation can be written

Acceler. Eq (4.49), $P = \sum w_i \varepsilon_i$ (5.5) $\Rightarrow$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \sum_{i=1}^{N} \underbrace{\varepsilon_i(1 + 3w_i)}_{(\varepsilon + 3P)}. \tag{6.7}$$

Divide each side of the acceleration equation by the square of the Hubble parameter $H(t)$ and change sign:

$$-\frac{\ddot{a}}{aH^2} = \frac{1}{2}\left[\frac{8\pi G}{3c^2H^2}\right] \sum_{i=1}^{N} \varepsilon_i(1 + 3w_i). \tag{6.8}$$

The quantity in square brackets in Equation 6.8 is the inverse of the critical energy density ε_c. Thus, we can rewrite the acceleration equation in the form

(4.30), (4.33) $\Rightarrow$

$$q_0 = -\frac{\ddot{a}}{aH^2} = \frac{1}{2} \sum_{i=1}^{N} \Omega_i(1 + 3w_i). \tag{6.9}$$

Evaluating Equation 6.9 at the present moment, $t = t_0$, tells us the relation between the deceleration parameter q_0 and the density parameters of the different components of the universe:

$$q_0 = \frac{1}{2} \sum_{i=1}^{N} \Omega_{i,0}(1 + 3w_i). \tag{6.10}$$

For a universe containing radiation ($w = \frac{1}{3}$), matter ($w = 0$), and a cosmological constant ($w = -1$),

$9 \times 10^{-5} + \frac{1}{2} \times 0.32 - 0.68 \Rightarrow q_0 \simeq -0.52$

$$\boxed{q_0 = \Omega_{r,0} + \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0}.} \tag{6.11}$$

Such a universe will currently be accelerating outward ($q_0 < 0$) if $\Omega_{\Lambda,0} > \Omega_{r,0} + \Omega_{m,0}/2$. The Benchmark Model, for instance, has $q_0 \approx -0.53$. (-0.52)

In principle, determining H_0 should be easy. For small redshifts, the relation between a galaxy's distance d and its redshift z is linear Equation (2.8):

$$\sim \; cz = H_0 d. \tag{6.12}$$

Thus, if you measure the distance d and redshift z for a large sample of galaxies, and fit a straight line to a plot of cz versus d, the slope of the plot gives you the value of H_0.[1] In practice, the distance to a galaxy is not only difficult to measure,

[1] The peculiar velocities of galaxies cause a significant amount of scatter in the plot, but by using a large number of galaxies, you can beat down the statistical errors. If you use galaxies at $d < 100\,\text{Mpc}$, you must also make allowances for the local inhomogeneity and anisotropy.

but also somewhat difficult to define. In Section 3.5, the proper distance $d_p(t)$ between two points was defined as the length of the spatial geodesic between the points when the scale factor is fixed at the value $a(t)$. The proper distance is perhaps the most straightforward definition of the spatial distance between two points in an expanding universe. Moreover, there is a helpful relation between scale factor and proper distance. If we observe, at time t_0, light that was emitted by a distant galaxy at time t_e, the current proper distance to that galaxy is (Equation 5.33):

(5.33) $\Leftrightarrow$
$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)}. \tag{6.13}$$

For the model universes examined in Chapter 5, we knew the exact functional form of $a(t)$, and hence could exactly compute $d_p(t_0)$ for a galaxy of any redshift. If we have only partial knowledge of the scale factor, in the form of the Taylor expansion of Equation 6.4, we may use the expansion

$$(6.4) \Longrightarrow (1+z) = \frac{1}{a(t)} \approx 1 - H_0(t-t_0) + \left(1+\frac{q_0}{2}\right)H_0^2(t-t_0)^2 \tag{6.14}$$

in Equation 6.13. Including the two lowest-order terms in the lookback time, $t_0 - t_e$, we find that the proper distance to the galaxy is

$$d_p(t_0) \approx \underbrace{c(t_0-t_e)}_{\text{static comp.}} + \overbrace{\frac{cH_0}{2}(t_0-t_e)^2}^{\text{expanding comp}} + \ldots \tag{6.15}$$

The first term in the above equation, $c(t_0 - t_e)$, is what the proper distance would be in a static universe – the lookback time times the speed of light. The second term is a correction due to the expansion of the universe during the time the light was traveling.

Equation 6.15 would be extremely useful if the photons from distant galaxies carried a stamp telling us the lookback time, $t_0 - t_e$. They don't; instead, they carry a stamp telling us the scale factor $a(t_e)$ at the time the light was emitted. The observed redshift z of a galaxy, remember, is

$$a(t_e) \equiv \frac{1}{(1+z)} \Rightarrow \qquad z = \frac{1}{a(t_e)} - 1. \tag{6.16}$$

Using Equation 6.14, we may write an approximate relation between redshift and lookback time: $z \simeq H_0(t_0-t_e)\left[1+\left(1+\frac{q_0}{2}\right)H_0(t_0-t_e)\right]$

$$\left.\begin{matrix}(6.14)\\(6.16)\end{matrix}\right\} \Rightarrow \qquad z \approx H_0(t_0-t_e) + \left(1+\frac{q_0}{2}\right)H_0^2(t_0-t_e)^2. \tag{6.17}$$

Inverting Equation 6.17 to give the lookback time as a function of redshift, we find

$\boxed{\text{H/W 7.1}}$ * Integrate (6.13) with (6.14) to derive (6.15)

$$\boxed{t_0 - t_e \approx H_0^{-1}\left[z - \left(1+\frac{q_0}{2}\right)z^2\right].} \tag{6.18}$$

* PROVE THAT (6.18) SOLVES (6.17)

* SHOW THAT (6.19) FOLLOWS: $d_p(t_0) \simeq \frac{cz}{H_0}\left[1-\left(\frac{1+q_0}{2}\right)z\right] \simeq R_0 z(1-0.24z)$ for $z \ll 4.2$

Substituting Equation 6.18 into Equation 6.15 gives us an approximate relation for the current proper distance to a galaxy with redshift z:

$$d_p(t_0) \approx \frac{c}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right)z^2\right] + \frac{cH_0}{2}\frac{z^2}{H_0^2} = \boxed{\frac{c}{H_0}z\left[1 - \frac{1+q_0}{2}z\right]} = d_p(t_0)(z) \qquad (6.19)$$

The linear Hubble relation $d_p \propto z$ thus holds true only in the limit $z \ll 2/(1+q_0) \simeq \frac{2}{1-0.52} \simeq 4.2$ ($z \ll 4.2$). If $q_0 > -1$, then the proper distance to a galaxy of moderate redshift ($z \sim 0.1$, say) is less than would be predicted from the linear Hubble relation.

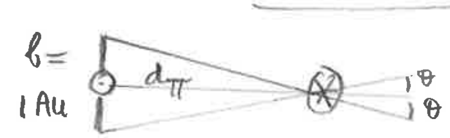
6.2 Luminosity Distance

Unfortunately, the current proper distance to a galaxy, $d_p(t_0)$, is not a measurable property. If you tried to measure the distance to a galaxy with a tape measure, for instance, the distance would be continuously increasing as you extended the tape. To measure the proper distance at time t_0, you would need a tape measure that could be extended with infinite speed; alternatively, you would need to stop the expansion of the universe at its current scale factor while you measured the distance at your leisure. Neither of these alternatives is physically possible.

Since cosmology is ultimately based on observations, if we want to find the distance to a galaxy, we need some way of computing a distance from that galaxy's observed properties. In devising ways of computing the distance to galaxies, astronomers have found it useful to adopt and adapt the techniques used to measure shorter distances. Let's examine, then, the techniques used to measure relatively short distances. Within the solar system, astronomers measure the distance to planets by reflecting radar signals from them. If δt is the time taken for a photon to complete the round-trip, then the distance to the reflecting body is $d = c\,\delta t/2$. (Since the relative speeds of objects within the solar system are much smaller than c, the corrections due to relative motion during the time δt are minuscule.) The accuracy with which distances have been determined with this technique is impressive; the length of the astronomical unit, for instance, is $\boxed{1\,\text{AU} = 149\,597\,870.7\,\text{km}}$. The radar technique is useful only within the solar system. Beyond ~ 10 AU, the reflected radio waves are too faint to detect.

A favorite method for determining distances to other stars within our galaxy is the method of trigonometric parallax. When a star is observed from two points separated by a distance b, the star's apparent position will shift by an angle θ. If the baseline of observation is perpendicular to the line of sight to the star, the *parallax distance* will be

SAA: $\theta = \frac{b}{d_\pi} \Rightarrow \frac{d_\pi}{pc} = \frac{b/AU}{\theta/arcsec}$



$$d_\pi = 1\,\text{pc}\left(\frac{b}{1\,\text{AU}}\right)\left(\frac{\theta}{1\,\text{arcsec}}\right)^{-1} \qquad (6.20)$$

Measuring the distances to stars using the Earth's orbit ($b = 2$ AU) as a baseline is a standard technique. Since the size of the Earth's orbit is known with great accuracy from radar measurements, the accuracy with which the parallax distance can be determined is limited by the accuracy with which the parallax angle θ can be measured. The *Gaia* satellite, launched by the European Space Agency in 2013, was designed to measure the parallax of stars with an error as small as ~ 10 microarcseconds. However, to measure θ for a galaxy 100 Mpc away, an error of < 0.01 microarcseconds would be required, using the Earth's orbit as a baseline. The trigonometric parallaxes of galaxies at cosmological distances are too small to be measured with current technology.

Let's focus on the properties that we *can* measure for objects at cosmological distances. We can measure the flux of light, f, from the object, in units of watts per square meter. The complete flux, integrated over all wavelengths of light, is called the *bolometric* flux. The adjective "bolometric" is a reference to the scientific instrument known as a bolometer, an extremely sensitive thermometer capable of detecting electromagnetic radiation over a wide range of wavelengths. The bolometer was invented around the year 1880 by the astronomer Samuel Langley, who used it to measure solar radiation. As expressed more poetically in an anonymous limerick:

Oh, Langley devised the bolometer:
It's really a kind of thermometer
Which measures the heat
From a polar bear's feet
At a distance of half a kilometer.[2]

More prosaically, given the technical difficulties of measuring the true bolometric flux, the flux over a limited range of wavelengths is measured. If the light from the object has emission or absorption lines, we can measure the redshift, z. If the object is an extended source rather than a point of light, we can measure its angular diameter, $\delta\theta$.

One way of using measured properties to assign a distance is the *standard candle* method. A standard candle is an object whose luminosity L is known. For instance, if some class of astronomical object had luminosities that were the same throughout all of spacetime, they would act as excellent standard candles – if their unique luminosity L were known. If you know, by some means or other, the luminosity of an object, then you can use its measured flux f to compute a function known as the *luminosity distance*, defined as

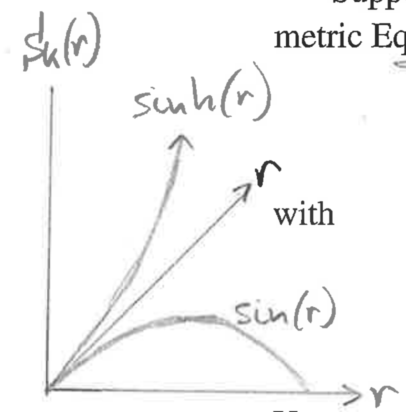
INV. SQ LAW: $f_\nu = \frac{L_\nu}{4\pi d_L^2}$

$$d_L \equiv \left(\frac{L}{4\pi f_\nu}\right)^{1/2}. \tag{6.21}$$

[2] The earliest version of this poem that I can find (in the May 1950 issue of *Electronics*) refers to the polar bear's seat, rather than its feet. I leave it to you to choose your favorite bit of the bear's anatomy.

The function d_L is called a "distance" because its dimensionality is that of a distance, and because it is what the proper distance to the standard candle would be *if* the universe were static and Euclidean. In a static Euclidean universe, propagation of light follows the inverse square law: $f = L/(4\pi d^2)$.

Suppose, though, that you are in a universe described by a Robertson–Walker metric Equation (3.41):



$$ds^2 = -c^2dt^2 + \overbrace{a(t)^2[dr^2 + S_\kappa(r)^2 d\Omega^2]}^{\text{proper distance}}, \tag{6.22}$$

with

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & (\kappa = +1) \\ r & (\kappa = 0) \\ R_0 \sinh(r/R_0) & (\kappa = -1). \end{cases} \tag{6.23}$$

You are at the origin. At the present moment, $t = t_0$, you see light that was emitted by a standard candle at comoving coordinate location (r, θ, ϕ) at a time t_e (Figure 6.1). The photons emitted at time t_e are, at the present moment, spread over a sphere of proper radius $d_p(t_0) = r$ and proper surface area $A_p(t_0)$. If space is flat ($\kappa = 0$), then the proper area of the sphere is given by the Euclidean relation $A_p(t_0) = 4\pi d_p(t_0)^2 = 4\pi r^2$. More generally, however,

$$A_p(t_0) = 4\pi S_\kappa(r)^2. \tag{6.24}$$

When space is positively curved, $A_p(t_0) < 4\pi r^2$, and the photons are spread over a smaller area than they would be in flat space. When space is negatively curved, $A_p(t_0) > 4\pi r^2$, and photons are spread over a larger area than they would be in flat space.

In addition to these geometric effects, which apply even in a static universe, the expansion of the universe causes the observed flux of light from a standard candle of redshift z to be decreased by a factor of $(1 + z)^{-2}$. First, the expansion of the universe causes the energy of each photon from the standard candle to decrease. If a photon starts with an energy $E_e = hc/\lambda_e$ when the scale factor is

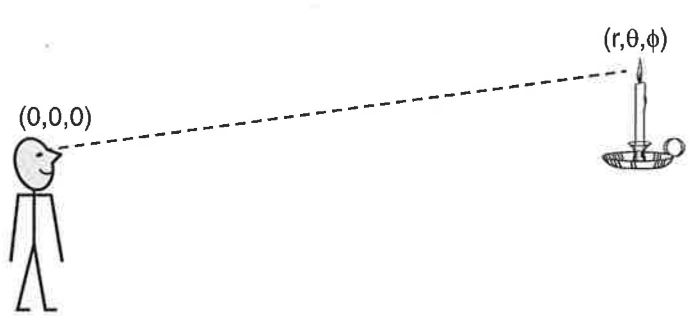


Figure 6.1 An observer at the origin observes a standard candle, of known luminosity L, at comoving coordinate location (r, θ, ϕ).

$a(t_e)$, by the time we observe it, when the scale factor is $a(t_0) = 1$, the wavelength will have grown to

$\frac{\lambda_0}{\lambda_e} = (1+z) \Longrightarrow$

$$\frac{c}{\nu_0} = \lambda_0 = \frac{1}{a(t_e)}\lambda_e = (1+z)\lambda_e = (1+z)\cdot\frac{c}{\nu_e} \tag{6.25}$$

and the energy will have fallen to

$E_e = h\nu_e = \frac{hc}{\lambda_e}$

$$\boxed{E_0 = \frac{E_e}{1+z}.} \tag{6.26}$$

Second, thanks to the expansion of the universe, the time between photon detections will be greater. If two photons are emitted in the same direction separated by a time interval δt_e, the proper distance between them will initially be $c(\delta t_e)$; by the time we detect the photons at time t_0, the proper distance between them will be stretched to $c(\delta t_e)(1+z)$, and we will detect them separated by a time interval $\delta t_0 = \delta t_e(1+z)$. $\Longleftrightarrow$ (Special) Relativistic Time Dilation!

The net result is that in an expanding, spatially curved universe, the relation between the observed flux f and the luminosity L of a distant light source is

$$f = \frac{L}{4\pi S_\kappa(r)^2(1+z)^2}, \tag{6.27}$$

and the luminosity distance is

$$\boxed{d_L = S_\kappa(r)(1+z).} \tag{6.28}$$

$\Longleftrightarrow f = \frac{L}{4\pi d_L^2}$

In V Ryden

$S_k(r) = r = d_p(t_0)$ for $k=0$

The available evidence indicates that our universe is nearly flat, with a radius of curvature R_0 much larger than the current horizon distance $d_{\rm hor}(t_0)$. Objects with finite redshift are at proper distances smaller than the horizon distance, and hence much smaller than the radius of curvature. Thus, it is safe to make the approximation $r \ll R_0$, implying $S_\kappa(r) \approx r$. With our assumption that space is very close to being flat, the relation between the luminosity distance and the current proper distance becomes very simple:

$$\boxed{d_L = r(1+z) = d_p(t_0)(1+z)} \qquad [\kappa = 0]. \tag{6.29}$$

Thus, even if space is perfectly flat, if you estimate the distance to a standard candle by using a naïve inverse square law, you will overestimate the actual proper distance by a factor $(1+z)$, where z is the standard candle's redshift.

Figure 6.2 shows the luminosity distance d_L as a function of redshift for the Benchmark Model and for two other flat universes, one dominated by matter and one dominated by a cosmological constant. When $z \ll 1$, the current proper distance may be approximated as

[HW.7] (6.19) $\Longrightarrow$

$$\boxed{d_P(t_0) \approx \frac{c}{H_0}z\left(1 - \frac{1+q_0}{2}z\right).} \tag{6.30}$$

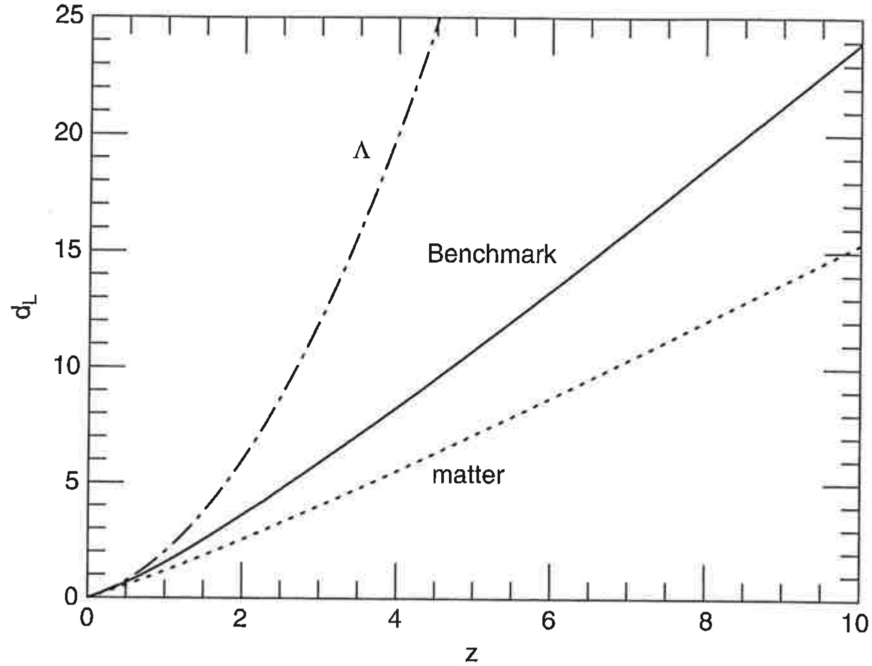


Figure 6.2 Luminosity distance of a standard candle with observed redshift z, in units of the Hubble distance, c/H_0. The bold solid line gives the result for the Benchmark Model. For comparison, the dot-dash line indicates a flat, lambda-only universe, and the dotted line a flat, matter-only universe.

In a nearly flat universe, the luminosity distance may thus be approximated as

$$d_L \approx \frac{c}{H_0} z \left(1 - \frac{1+q_0}{2} z\right)(1+z) \approx \boxed{\frac{c}{H_0} z \left(1 + \frac{1-q_0}{2} z\right) = d_L} \qquad (6.31)$$

So that, of course, for $z \to 0$: $d_L(z) = d_p(t_0) = d_A \simeq \frac{c}{H_0} z \simeq R_0 z$!

6.3 Angular-diameter Distance

The luminosity distance d_L is not the only distance measure that can be computed using the observable properties of cosmological objects. Suppose that instead of a standard candle, you observed a *standard yardstick*. A standard yardstick is an object whose proper length ℓ is known. In many cases, it is convenient to choose as your yardstick an object that is tightly bound together, by gravity or duct tape or some other influence, and hence is not expanding along with the universe as a whole.

Suppose a yardstick of constant proper length ℓ is aligned perpendicular to your line of sight, as shown in Figure 6.3. You measure an angular distance $\delta\theta$ between the ends of the yardstick, and a redshift z for the light that the yardstick emits. If $\delta\theta \ll 1$, and if you know the length ℓ of the yardstick, you can compute a distance to the yardstick using the small-angle formula

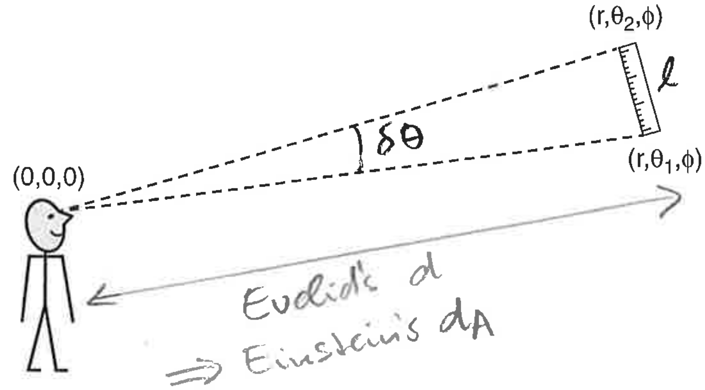


Figure 6.3 An observer at the origin observes a standard yardstick, of known proper length ℓ, at comoving coordinate distance r.

Euclid's SAA:

$$\delta\theta = \frac{\ell}{d} = \underbrace{\frac{\ell}{d_A}}_{\text{Einstein SAA}} \Longrightarrow \boxed{d_A \equiv \frac{\ell}{\delta\theta}.} \qquad (6.32)$$

This function of ℓ and $\delta\theta$ is called the *angular-diameter distance*. The angular-diameter distance is equal to the proper distance to the yardstick if the universe is static and Euclidean.

In general, though, if the universe is expanding or curved, the angular-diameter distance will not be equal to the current proper distance. Suppose you are in a universe described by the Robertson–Walker metric given in Equation 6.22. Choose your comoving coordinate system so that you are at the origin. The yardstick is at a comoving coordinate distance r. At a time t_e, the yardstick emitted the light that you observe at time t_0. The comoving coordinates of the two ends of the yardstick, at the time the light was emitted, were (r, θ_1, ϕ) and (r, θ_2, ϕ). As the light from the yardstick moves toward the origin, it travels along geodesics with $r =$ constant and $\phi =$ constant. Thus, the angular size you measure for the yardstick will be $\delta\theta = \theta_2 - \theta_1$. The distance ds between the two ends of the yardstick, measured at the time t_e when the light was emitted, can be found from the Robertson–Walker metric:

TYPO!

geodesics $dr = 0, d\varphi = 0$

(3.33) $ds^2 = dr^2 + S_k(r)^2 . d\Omega^2$

(3.34) $d\Omega^2 = d\theta^2 + \sin^2\theta . d\varphi^2$

with $dr \equiv 0, d\varphi \equiv 0$

$$\Longrightarrow ds = a(t_e)S_\kappa(r)\delta\theta. \Longleftrightarrow d\theta = \frac{ds}{a(t_e).S_k(r)} \simeq \frac{ds}{a(t_e).r} \text{ for } k=0 \qquad (6.33)$$

However, for a standard yardstick whose length ℓ is known, we can set $ds = \ell$, and thus find that

$$\ell = a(t_e)S_\kappa(r)\delta\theta = \frac{S_\kappa(r)\delta\theta}{1+z} \overset{\text{Einstein}}{\equiv} d_A . \delta\theta \Longleftarrow (6.32) \qquad (6.34)$$

Thus, the angular-diameter distance d_A to a standard yardstick is

$$\boxed{d_A \equiv \frac{\ell}{\delta\theta} = \frac{S_\kappa(r)}{1+z} = \frac{d_p(t_0)}{(1+z)} = \frac{d_L}{(1+z)^2}} \qquad (6.35)$$

Einstein's Small Angle Approximation

Comparison with Equation 6.28 shows that the relation between the angular-diameter distance and the luminosity distance is

$$\boxed{d_A = \frac{d_L}{(1+z)^2}.} \tag{6.36}$$

Thus, if you observe a redshifted object that is both a standard candle and a standard yardstick, the angular-diameter distance that you compute for the object will be smaller than the luminosity distance. Moreover, if the universe is spatially flat,

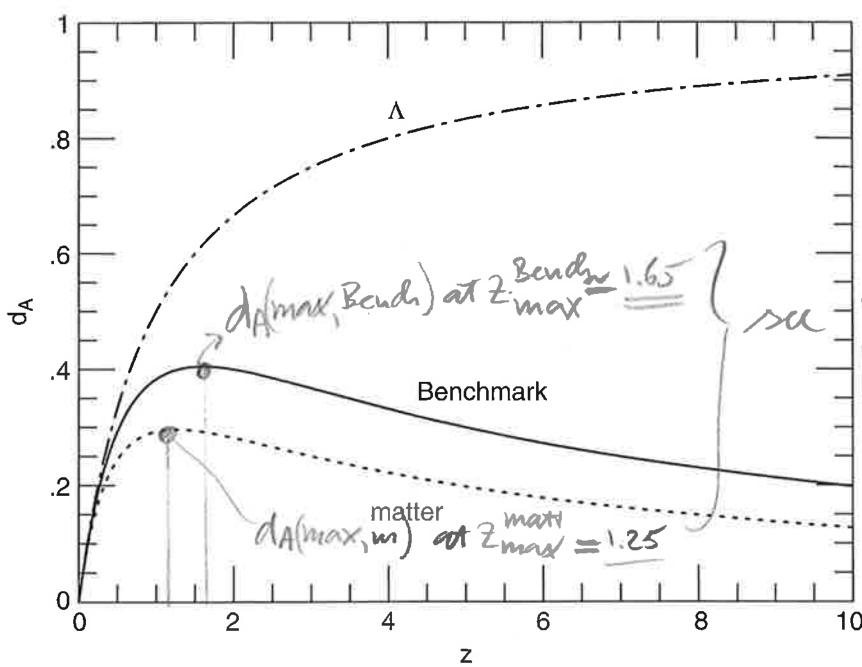
TO PRESERVE BOTH SAA + INV. SQ LAW, YOU MUST HAVE:

$$\boxed{d_A(1+z) = d_p(t_0) = \frac{d_L}{1+z}} \qquad [\kappa = 0]. \tag{6.37}$$

In a flat universe, therefore, if you compute the angular-diameter distance d_A of a standard yardstick, it isn't equal to the current proper distance $d_p(t_0)$; rather, it is equal to the proper distance at the time the light from the object was emitted: $\boxed{d_A = d_p(t_0)/(1+z) = d_p(t_e).}$

Figure 6.4 shows the angular-diameter distance d_A for the Benchmark Model, and for two other spatially flat universes, one dominated by matter and one dominated by a cosmological constant. [Since d_A is, for these flat universes, equal to $d_p(t_e)$, Figure 6.4 is simply a replotting of the right panel in Figure 5.9.] When $z \ll 1$, the approximate value of d_A is given by the expansion

but on a linear scale to better see details!



see Fig 5.9b $\boxed{\text{H/W 5.6}}$ or Team Project Honors

Figure 6.4 Angular-diameter distance of a standard yardstick with observed redshift z, in units of the Hubble distance, c/H_0. The bold solid line gives the result for the Benchmark Model. For comparison, the dot-dash line indicates a flat, lambda-only universe, and the dotted line a flat, matter-only universe.

$$(6.30)\ d_p(t_0) \simeq \frac{c}{H_0} z\left(1-\frac{1+q_0}{2}z\right) \Rightarrow d_A \approx \frac{c}{H_0} z\left(1-\frac{3+q_0}{2}z\right). \qquad (6.38)$$

For: $z \to 0$

Thus, comparing Equations 6.30, 6.31, and 6.38, we find that in the limit $z \to 0$, $d_A \approx d_L \approx d_p(t_0) \approx (c/H_0)z$. However, the state of affairs is very different in the limit $z \to \infty$. In models with a finite horizon size, $d_p(t_0) \to d_{\rm hor}(t_0)$ as $z \to \infty$. The luminosity distance to highly redshifted objects, in this case, diverges as $z \to \infty$, with

$$d_L(z \to \infty) \approx z d_{\rm hor}(t_0). \to \infty \text{ for } z \to \infty\ ! \qquad (6.39)$$

However, the angular-diameter distance to highly redshifted objects approaches zero as $z \to \infty$, with

$$d_A = \frac{d_L}{(1+z)^2} = \frac{z\, d_{\rm hor}(t_0)}{(1+z)^2} \Rightarrow \qquad d_A(z \to \infty) \approx \frac{d_{\rm hor}(t_0)}{z}. \to 0 \text{ for } z \to \infty\ ! \qquad (6.40)$$

In model universes other than the lambda-only model, the angular-diameter distance d_A has a maximum for standard yardsticks at some critical redshift z_c. For instance, the Benchmark Model has a critical redshift $z_c = 1.6$, where $d_A(\max) = 0.405c/H_0 = 1770\,\text{Mpc}$. If the universe were full of glow-in-the-dark yardsticks, all of the same size ℓ, their angular size $\delta\theta$ would decrease with redshift out to $z = z_c$, but then would linearly increase at larger redshifts. !!! The sky would be full of big, faint, redshifted yardsticks.

In principle, standard yardsticks, like standard candles, can be used to measure cosmological parameters such as H_0, $\Omega_{\Lambda,0}$, and $\Omega_{m,0}$. In practice, the use of standard yardsticks to determine cosmological parameters was long plagued with observational difficulties. For instance, a standard yardstick must have an angular size large enough to be resolved by your telescope. A yardstick of physical size ℓ will have its angular size $\delta\theta$ minimized when it is at the critical redshift z_c. For the Benchmark Model, 1 kpc at $z \simeq 1$–$2 \Leftrightarrow \delta\theta \simeq 0.12''$ or $\delta\theta \simeq 1''$ for $\ell \simeq 8.6$ kpc at $z \simeq 1$–2.

$$\delta\theta(\min) = \frac{\ell}{d_A(\max)} = \frac{\ell}{1770\,\text{Mpc}} \approx 0.12\,\text{arcsec}\left(\frac{\ell}{1\,\text{kpc}}\right). \qquad (6.41)$$

(see Ned Wright URL on Links)

Both galaxies and clusters of galaxies are large enough to be useful standard candles. Unfortunately for cosmologists, galaxies and clusters of galaxies do not have sharply defined edges, so assigning a particular angular size $\delta\theta$, and a corresponding physical size ℓ, to these objects is a somewhat tricky task. Moreover, galaxies and clusters of galaxies are not isolated, rigid yardsticks of fixed length. Galaxies tend to become larger with time as they undergo mergers with their neighbors. Clusters, too, tend to become larger with time, as galaxies fall into them. Correcting for these evolutionary trends is a difficult task. Given the historical difficulties involved in using standard yardsticks to determine cosmological parameters, let's first look at how standard *candles* can be used to determine H_0.

6.4 Standard Candles and H_0

Using standard candles to determine the Hubble constant has a long and honorable history; it's the method used by Hubble himself. The recipe for finding the Hubble constant is a simple one:

- Identify a population of standard candles with luminosity L.
- Measure the redshift z and flux f for each standard candle.
- Compute $d_L = (L/4\pi f)^{1/2}$ for each standard candle.
- Plot cz versus d_L.
- Measure the slope of the cz versus d_L relation when $z \ll 1$; this gives H_0.

As with the apocryphal recipe for rabbit stew that begins "First catch your rabbit," the hardest step is the first one. A good standard candle is hard to find. For cosmological purposes, a standard candle should be bright enough to be detected at large redshifts. It should also have a luminosity that is well determined.[3]

One time-honored variety of standard candle is the class of *Cepheid variable stars*. Cepheids, as they are known, are highly luminous supergiant stars, with mean luminosities in the range $\bar{L} = 400 \rightarrow 40\,000\,\mathrm{L}_\odot$. Cepheids are pulsationally unstable. As they pulsate radially, their luminosity varies in response, partially due to the changes in their surface area, and partially due to the changes in the surface temperature as the star pulsates. The pulsational periods, as reflected in the observed brightness variations of the star, lie in the range $P = 1.5 \rightarrow 60$ days.

On the face of it, Cepheids don't seem sufficiently standardized to be standard candles; their mean luminosities range over two orders of magnitude. How can you tell whether you are looking at an intrinsically faint Cepheid ($L \approx 400\,\mathrm{L}_\odot$) or at an intrinsically bright Cepheid ($L \approx 40\,000\,\mathrm{L}_\odot$) ten times farther away? The key to calibrating Cepheids was discovered by Henrietta Leavitt, at Harvard College Observatory. In the years prior to World War I, Leavitt was studying variable stars in the Large and Small Magellanic Clouds, a pair of relatively small satellite galaxies orbiting our own galaxy. For each Cepheid in the Small Magellanic Cloud (SMC), she measured the period P by finding the time between maxima in the observed brightness, and found the mean flux $\bar{f}$, averaged over one complete period. She noted that there was a clear relation between P and $\bar{f}$, with stars having the longest period of variability also having the largest flux. Since the depth of the SMC, front to back, is small compared to its distance from us, she was justified in assuming that the difference in mean flux for the Cepheids was due to differences in their mean luminosity, not differences in their luminosity distance. Leavitt had discovered a period–luminosity relation for Cepheid variable stars.

[3] A useful cautionary tale in this regard is the saga of Edwin Hubble. In the 1929 paper that first demonstrated that $d_L \propto z$ when $z \ll 1$, Hubble underestimated the luminosity distances to galaxies by a factor of ~ 7 because he underestimated the luminosity of his standard candles by a factor of ~ 49.

If the same period–luminosity relation holds true for all Cepheids, in all galaxies, then Cepheids can act as a standard candle.

Suppose, for instance, you find a Cepheid star in the Large Magellanic Cloud (LMC) and another in M31. They both have a pulsational period of 10 days, so you assume, from the period–luminosity relation, that they have the same mean luminosity $\bar{L}$. By careful measurement, you determine that

$$\frac{\bar{f}_{\rm LMC}}{\bar{f}_{\rm M31}} = 230. \tag{6.42}$$

Thus, you conclude that the luminosity distance to M31 is greater than that to the LMC by a factor

$$\frac{d_L({\rm M31})}{d_L({\rm LMC})} = \left(\frac{\bar{f}_{\rm LMC}}{\bar{f}_{\rm M31}}\right)^{1/2} = \sqrt{230} = 15.2. \tag{6.43}$$

(In practice, given the intrinsic scatter in the period–luminosity relation, and the inevitable error in measuring fluxes, astronomers don't rely on a single Cepheid in each galaxy. Rather, they measure $\bar{f}$ and P for as many Cepheids as possible in each galaxy, then find the ratio of luminosity distances that makes the period–luminosity relations for the two galaxies coincide.)

Note that if you only know the relative fluxes of the two Cepheids, and not their luminosity $\bar{L}$, you will only know the *relative* distances of M31 and the LMC. To fix an absolute distance to M31, to the LMC, and to other galaxies containing Cepheids, you need to know the luminosity $\bar{L}$ for a Cepheid of a given period P. If, for instance, you could measure the parallax distance d_π to a Cepheid within our own galaxy, you could then compute its luminosity $\bar{L} = 4\pi d_\pi^2 \bar{f}$, and use it to normalize the period–luminosity relation for Cepheids.[4] Unfortunately, Cepheids are rare stars; only the very nearest Cepheids in our galaxy have had their distances measured accurately. The nearest Cepheid is Polaris, as it turns out, at $d_\pi = 130 \pm 10\,{\rm pc}$. The next nearest is δ Cephei (the prototype after which all Cepheids are named), at $d_\pi = 270 \pm 10\,{\rm pc}$. Historically, given the lack of Cepheid parallaxes, astronomers have relied on alternative methods of normalizing the period–luminosity relation for Cepheids. The most usual method involved finding the distance to the Large Magellanic Cloud by secondary methods, then using this distance to compute the mean luminosity of the LMC Cepheids. The current consensus is that the Large Magellanic Cloud has a luminosity distance $d_L = 50 \pm 2\,{\rm kpc}$, implying a distance to M31 of $d_L = 760 \pm 30\,{\rm kpc}$.

$d_{LMC} \simeq 50$ kpc ± 2

$d_{M31} \simeq 15.2\, d_{LMC}$

$d_{M31} \simeq 760$ kpc ± 30

$d_{LMC} =$

The fluxes and periods of Cepheids can be accurately measured out to luminosity distances $d_L \sim 30\,{\rm Mpc}$. Observation of Cepheid stars in the Virgo cluster of galaxies, for instance, has yielded a distance $d_L({\rm Virgo}) = 300\, d_L({\rm LMC})$ = 15 Mpc

[4] Within our galaxy, which is not expanding, the parallax distance, the luminosity distance, and the proper distance are identical.

15 Mpc. One of the motivating reasons for building the *Hubble Space Telescope* in the first place was to use Cepheids to determine H_0. The net result of the Hubble Key Project to measure H_0 is displayed in Figure 2.5, showing that the Cepheid data are best fitted with a Hubble constant of $H_0 = 75 \pm 8\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. LATEST $H_0 \simeq 73.6$ (Riess et al 2011)

There is a hidden difficulty involved in using Cepheid stars to determine H_0. Cepheids can take you out only to a distance $d_L \sim 30\,\mathrm{Mpc}$; on this scale, the universe cannot be assumed to be homogeneous and isotropic. In fact, the Local Group is gravitationally attracted toward the Virgo cluster, causing it to have a peculiar motion in that direction. It is estimated, from dynamical models, that the recession velocity cz that we measure for the Virgo cluster is $250\,\mathrm{km\,s^{-1}}$ less than it would be if the universe were perfectly homogeneous. The plot of cz versus d_L given in Figure 2.5 uses recession velocities that are corrected for this "Virgocentric flow," as it is called.

6.5 Standard Candles and Acceleration

$z_{\text{coma cl}} \simeq 0.023$

To determine the value of H_0 without having to worry about Virgocentric flow and other peculiar velocities, we need to determine the luminosity distance to standard candles with $d_L > 100\,\mathrm{Mpc}$, or $z > 0.02$. To determine the acceleration of the universe, we need to view standard candles for which the relation between d_L and z deviates significantly from the linear relation that holds true at lower redshifts. In terms of H_0 and q_0, the luminosity distance at small redshift is, from Equation 6.31,

$$d_L \approx \frac{c}{H_0} z \left[1 + \frac{1 - q_0}{2} z\right]. \tag{6.44}$$

At a redshift $z = 0.2$, for instance, the luminosity distance d_L in the Benchmark Model (with $q_0 = -0.53$) is 5 percent larger than d_L in an empty universe (with $q_0 = 0$).

For a standard candle to be seen at $d_L > 1000\,\mathrm{Mpc}$, it must be very luminous. In recent years, the standard candle of choice among cosmologists has been *type Ia supernovae*. A supernova may be loosely defined as an exploding star. Early in the history of supernova studies, when little was known about their underlying physics, supernovae were divided into two classes, on the basis of their spectra. Type I supernovae contain no hydrogen absorption lines in their spectra; type II supernovae contain strong hydrogen absorption lines. Gradually, it was realized that all type II supernovae are the same species of beast; they are massive stars ($M > 8\,\mathrm{M_\odot}$) whose cores collapse to form a black hole or neutron star when their nuclear fuel is exhausted. During the rapid collapse of the core, the outer layers of the star are thrown off into space. Type I supernovae are actually two separate species, called type Ia and type Ib. Type Ib supernovae, it is thought, are massive stars whose cores collapse after the hydrogen-rich outer layers of the star have been blown away in strong stellar winds. Thus, type Ib and type II supernovae are driven by very similar mechanisms – their differences are superficial, in the most literal sense. Type Ia supernovae, however, are something completely different. They begin as white dwarfs; that is, stellar remnants that are supported against gravity by the quantum mechanical effect known as electron degeneracy pressure. The maximum mass at which a white dwarf can be supported against its self-gravity is called the Chandrasekhar mass; the value of the Chandrasekhar mass is $M \approx 1.4\,\mathrm{M}_\odot$. A white dwarf can go over this limit by merging with another white dwarf, or by accreting gas from a stellar companion. If the Chandrasekhar limit is approached or exceeded, the white dwarf starts to collapse until its increased density triggers a runaway nuclear fusion reaction. The entire white dwarf becomes a fusion bomb, blowing itself to smithereens; unlike type II supernovae, type Ia supernovae do not leave a condensed stellar remnant behind.

Within our galaxy, type Ia supernovae occur roughly once per century, on average. Although type Ia supernovae are not frequent occurrences locally, they are extraordinarily luminous, and hence can be seen to large distances. The luminosity of an average type Ia supernova, at peak brightness, is $L = 4 \times 10^9\,\mathrm{L}_\odot$; $\simeq 1.53 \times 10^{36}$ J/sec! that's 100 000 times more luminous than even the brightest Cepheid. For a few $= 1.5 \times 10^{43}$ ergs/s! days, a type Ia supernova in a moderately bright galaxy can outshine all the other stars in the galaxy combined. Since moderately bright galaxies can be seen at $z \sim 1$, this means that type Ia supernovae can also be seen at $z \sim 1$.

So far, type Ia supernovae sound like ideal standard candles; very luminous and all produced by the same mechanism. There's one complication, however. Observation of supernovae in galaxies whose distances have been well determined by Cepheids reveals that type Ia supernovae do not have identical luminosities. Instead of all having $L = 4 \times 10^9\,\mathrm{L}_\odot$, their peak luminosities lie in the fairly broad range $L \approx (3 \to 5) \times 10^9\,\mathrm{L}_\odot$. However, it has also been noted that the peak luminosity of a type Ia supernova is tightly correlated with the shape of its light curve. Type Ia supernovae with luminosities that shoot up rapidly and decline rapidly are less luminous than average at their peak; supernovae with luminosities that rise and fall in a more leisurely manner are more luminous than average. Thus, just as the period of a Cepheid tells you its luminosity, the rise and fall time of a type Ia supernova tells you its peak luminosity.

At the end of the 20th century, two research teams, the "Supernova Cosmology Project" and the "High-z Supernova Search Team," conducted searches for supernovae in distant galaxies, using the observed fluxes of the supernovae to constrain the acceleration of the expansion of the universe. To present the supernova results, I will have to introduce the "magnitude" system used by astronomers to express fluxes and luminosities. The magnitude system, like much else in astronomy, has its roots in ancient Greece. The Greek astronomer Hipparchus, in the second century BC, divided the stars into six classes, according to their apparent

1 mag = 2.51× factor

5 mag ≡ $10^2 \times = 2$ dex $= (2.51)^5$

brightness. The brightest stars were of "first magnitude," the faintest stars visible to the naked eye were of "sixth magnitude," and intermediate stars were ranked as second, third, fourth, and fifth magnitude. Long after the time of Hipparchus, it was realized that the response of the human eye is roughly logarithmic, and that stars of the first magnitude have fluxes (at visible wavelengths) about 100 times greater than stars of the sixth magnitude. On the basis of this realization, the magnitude system was placed on a more rigorous mathematical basis.

Nowadays, the bolometric *apparent magnitude* of a light source is defined in terms of the source's bolometric flux as

$f = \frac{L}{4\pi d_L^2}$

$$\boxed{m \equiv -2.5 \log_{10}(f/f_x),} \tag{6.45}$$

where the reference flux f_x is set at the value $f_x = 2.53 \times 10^{-8}$ watt m^{-2}. Thanks to the negative sign in the definition, a small value of m corresponds to a large flux f. For instance, the flux of sunlight at the Earth's location is $f = 1361$ watts m^{-2}; the Sun thus has a bolometric apparent magnitude of $m = -26.8$. The choice of reference flux f_x constitutes a tip of the hat to Hipparchus, since for stars visible to the naked eye it typically yields $0 < m < 6$.

The bolometric *absolute magnitude* of a light source is defined as the apparent magnitude that it would have if it were at a luminosity distance of $d_L = 10$ pc. Thus, a light source with luminosity L has a bolometric absolute magnitude

$L_x \equiv 78.7\, L_\odot$ at $d = d_L = 10$ pc

$\Updownarrow$

$f_x \equiv 2.53\ 10^{-8}$ W m^{-2}

$$\boxed{M \equiv -2.5 \log_{10}(L/L_x),} \tag{6.46}$$

$\Updownarrow$

$M_\odot = +4.74$ mag $(\equiv 2.5 \log 78.7)$

$m_\odot = -26.83$ mag

where the reference luminosity is $L_x = 78.7\, L_\odot$, since that is the luminosity of an object that produces a flux $f_x = 2.53 \times 10^{-8}$ watt m^{-2} when viewed from a distance of 10 parsecs. The bolometric absolute magnitude of the Sun is thus $M = 4.74$. Although the system of apparent and absolute magnitudes seems strange to the uninitiated, the apparent magnitude is really nothing more than a logarithmic measure of the flux, and the absolute magnitude is a logarithmic measure of the luminosity.

$m_\odot - M_\odot = 5 \log(d_\odot / 10\,\mathrm{pc})$

$\Downarrow$

$-26.83 - 4.74 = 5 \log d - 5$

$-31.57 = 5 \log d_\odot - 5$

$-26.57 = 5 \log d_\odot$

$\log d_\odot = -5.3144$ pc

$\Downarrow$

$d_\odot = 10^{-5.314}$ pc

$d_\odot = \frac{1\,\mathrm{pc}}{206{,}265}$ AU

$\Updownarrow$

$d_\odot \equiv 1$ AU

Given the definitions of apparent and absolute magnitude, the relation between an object's apparent magnitude and its absolute magnitude can be written in the form

log(INV. SA LA W) $\Rightarrow$

$$M = m - 5 \log_{10}\left(\frac{d_L}{10\,\mathrm{pc}}\right), \tag{6.47}$$

where d_L is the luminosity distance to the light source. If the luminosity distance is given in units of megaparsecs, this relation becomes

1 mag = 2.51×

$$\boxed{M = m - 5 \log_{10}\left(\frac{d_L}{1\,\mathrm{Mpc}}\right) - 25.\ \mathrm{mag}} \tag{6.48}$$

5 mag $= (2.51)^5 = 10^2 \times$

Since astronomers frequently quote fluxes and luminosities in terms of apparent and absolute magnitudes, they find it convenient to quote luminosity distances in

The supernova data extend out to $z \sim 1$; this is beyond the range where an expansion in terms of H_0 and q_0 is adequate to describe the scale factor $a(t)$. Thus, the two supernova teams customarily describe their results in terms of a model universe that contains both matter and a cosmological constant. After choosing values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, they compute the expected relation between $m - M$ and z, and compare it to the observed data. The results of fitting these model universes are given in Figure 7.6. The ovals drawn on Figure 7.6 enclose those values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ that give the best fit to the supernova data. The results of the two teams (the solid ovals and dotted ovals) give very similar results. Three concentric ovals are shown for each team's result; they correspond to 1σ, 2σ, and 3σ confidence intervals, with the inner oval representing the highest probability.

Slightly better (more instructive) fig 7.6 from Ryden 2003 1ST Edition, but with old, larger WMAP 2003 error ellipses.

We will get the best & smallest error ellipses from the Planck 2016 (see URL) and 2018 (forthcoming) papers

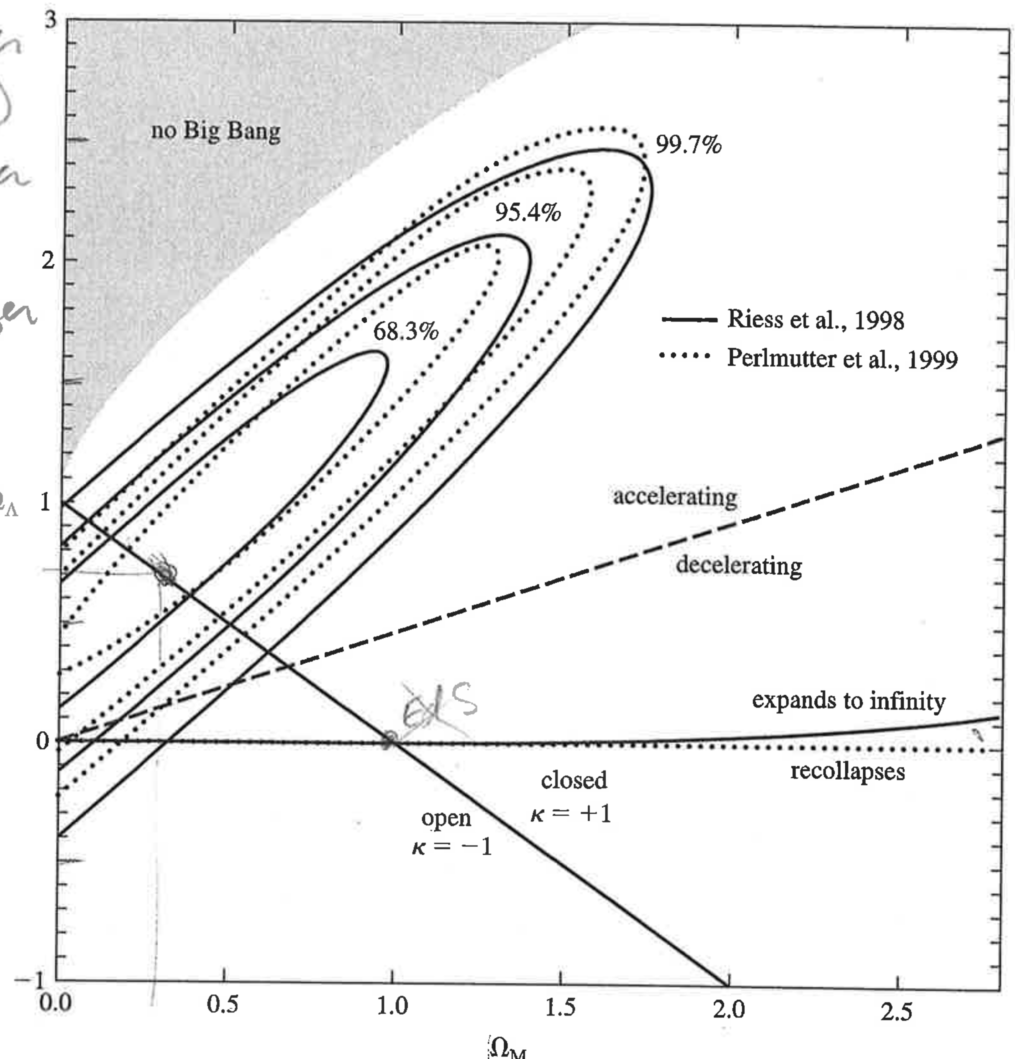


FIGURE 7.6 The values of $\Omega_{m,0}$ (horizontal axis) and $\Omega_{\Lambda,0}$ (vertical axis) that best fit the data shown in Figure 7.5. The solid ovals show the best-fitting values for the High-z Supernova Search Team data; the dotted ovals show the best-fitting values for the Supernova Cosmology Project data.

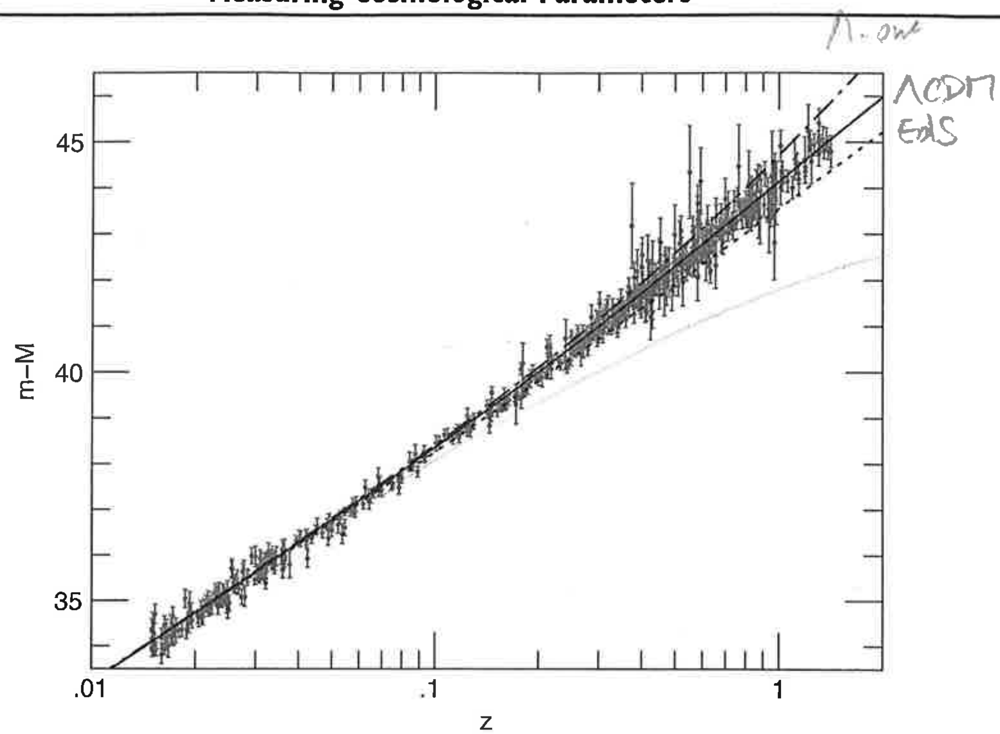


Figure 6.5 Distance modulus versus redshift for a set of 580 type Ia supernovae. The bold solid line gives the expected relation for the Benchmark Model. For comparison, the dot-dash line indicates a flat, lambda-only universe, and the dotted line a flat, matter-only universe. [data from Suzuki *et al.* 2012, *ApJ*, **716**, 85]

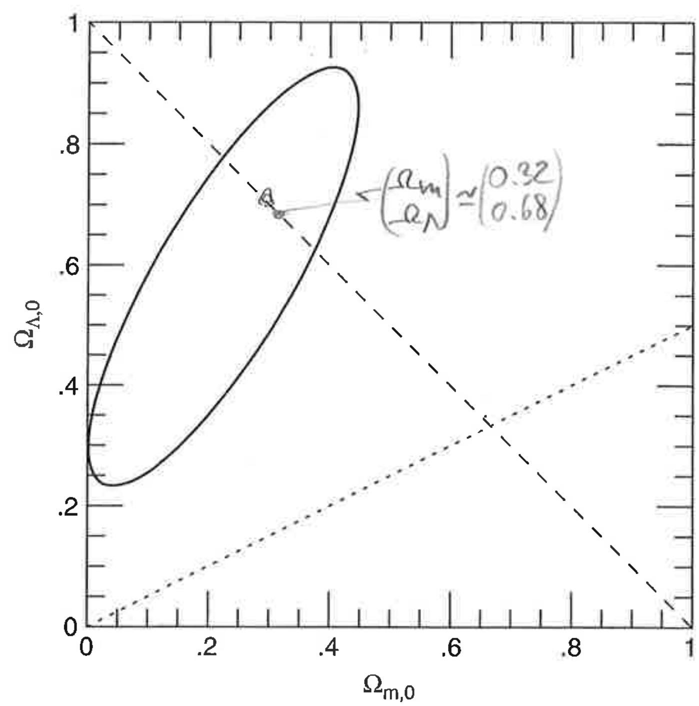


Figure 6.6 The values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ that best fit the supernova data. The bold elliptical contour represents the 95% confidence interval. For reference, the dashed line represents flat universes, and the dotted line represents coasting ($q_0 = 0$) universes: compare to Figure 5.6. [Anže Slosar & José Alberto Vázquez, Brookhaven National Laboratory]

terms of the *distance modulus* to a light source. The distance modulus is defined as $m - M$, and is related to the luminosity distance by the relation

$DM_{M31} \equiv (m-M)_{M31} \simeq 24.4$ mag
since $d_L^{(M31)} = 0.760$ Mpc

$$m - M = 5\log_{10}\left(\frac{d_L}{1\,\mathrm{Mpc}}\right) + 25. \tag{6.49}$$

Obj	d_L	$(m-M)$ mag
LMC	50 Kpc	18.5
M31	760	24.4
Virgo $z=.004$	15.0 Mpc	30.9
Coma $z=0.023$	80 Mpc	34.5
$z=1$	6.83 Gpc	44.2
$z=2$	16.0	46.0
$z=7$	70.6	49.2
$z=12$	Gpc 130.3	50.6

(see Windhorst 2018)

The distance modulus of the Large Magellanic Cloud, for instance, at $d_L = 0.050\,\mathrm{Mpc}$, is $(m - M)_{LMC} = 18.5$. The distance modulus of the Virgo cluster, at $d_L = 15\,\mathrm{Mpc}$, is $(m - M)_{VIR} = 30.9$. When $z \ll 1$, the luminosity distance to a light source is

$(6.31) \Rightarrow$

$$d_L \approx \frac{c}{H_0} z \left(1 + \frac{1 - q_0}{2} z\right). \tag{6.50}$$

Substituting this relation into Equation 6.49, we have an equation that gives the relation between distance modulus and redshift at low redshift:

H/w 22

$$\boxed{m - M \approx 43.23 - 5\log_{10}\left(\frac{H_0}{68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right) + 5\log_{10} z + 1.086(1 - q_0)z.} \tag{6.51}$$

* Show (6.51) from (6.50)

($H_0 = 66.9$ km/s/Mpc
$\Omega_m = 0.32$
$\Omega_\Lambda = 0.68$
$k=0$ ΛCDM)

* Prove table using (6.51) or Ned Wright URL

For a population of standard candles with known luminosity L (and hence of known bolometric absolute magnitude M), we measure the flux f (or equivalently the bolometric apparent magnitude m) and the redshift z. In the limit $z \to 0$, a plot of $m - M$ versus $\log z$ gives a straight line whose amplitude at a fixed z tells us the value of H_0. At slightly larger values of z, the deviation of the plot from a straight line tells us whether the expansion of the universe is speeding up or slowing down. At a given value of z, standard candles have a lower flux in an accelerating universe (with $q_0 < 0$) than in a decelerating universe (with $q_0 > 0$).

Figure 6.5 shows the plot of distance modulus versus redshift for a compilation of actual supernova observations from a variety of sources. The solid line running through the data is the result expected for the Benchmark Model. At a redshift $z \approx 1$, supernovae in the Benchmark Model are about 0.6 magnitudes fainter than they would be in a flat, matter-only universe; it was the observed faintness of Type Ia supernovae at $z > 0.3$ that led to the conclusion that the universe is accelerating. However, at $z \approx 1$, supernovae in the Benchmark Model are about 0.6 magnitudes brighter than they would be in a flat, lambda-only universe. Thus, the observations of Type Ia supernovae that tell us that the universe is accelerating also place useful upper limits on the magnitude of the acceleration.

Figure 6.6 shows the results of fitting the supernova data with different model universes; these models contain both matter and a cosmological constant, but are not required to be spatially flat. The bold ellipse represents the 95% confidence interval; that is, given the available set of supernova data, there is a 95% chance that the plotted ellipse contains the true values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. Notice from the plot that decelerating universes with $q_0 > 0$ (below the dotted line) are strongly excluded by the supernova data, as are Big Crunch universes and Big Bounce universes. However, the supernova data, taken by themselves, are consistent with positively curved or negatively curved universes, as well as with a flat universe. We will see in Chapter 8 how observations of the cosmic microwave background combine with the supernova results to suggest that we live in a universe that is both accelerating and spatially flat, with $\Omega_{m,0} \sim 0.3$ and $\Omega_{\Lambda,0} \sim 0.7$.

0.32 0.68 for $H_0 = 66.9$

Exercises

6.1 Suppose that a polar bear's foot has a luminosity of $L = 10$ watts. What is the bolometric absolute magnitude of the bear's foot? What is the bolometric apparent magnitude of the foot at a luminosity distance of $d_L = 0.5$ km? If a bolometer can detect the bear's foot at a maximum luminosity distance of $d_L = 0.5$ km, what is the maximum luminosity distance at which it could detect the Sun? What is the maximum luminosity distance at which it could detect a supernova with $L = 4 \times 10^9\, L_\odot$?

6.2 Suppose that a polar bear's foot has a diameter of $\ell = 0.16$ m. What is the angular size $\delta\theta$ of the foot at an angular-diameter distance of $d_A = 0.5$ km? In the Benchmark Model, what is the minimum possible angular size of the polar bear's foot?

6.3 Suppose that you are in a spatially flat universe containing a single component with a unique equation-of-state parameter w. What are the current proper distance $d_P(t_0)$, the luminosity distance d_L, and the angular-diameter distance d_A as a function of z and w? At what redshift will d_A have a maximum value? What will this maximum value be, in units of the Hubble distance?

6.4 Verify that Equation 6.51 is correct in the limit of small z. (You will probably want to use the relation $\log_{10}(1 + x) \approx 0.4343 \ln(1 + x) \approx 0.4343x$ in the limit $|x| \ll 1$.)

6.5 The surface brightness Σ of an astronomical object is its observed flux divided by its observed angular area; thus, $\Sigma \propto f/(\delta\theta)^2$. For a class of objects that are both standard candles and standard yardsticks, what is Σ as a function of redshift? Would observing the surface brightness of this class of objects be a useful way of determining the value of the deceleration parameter q_0? Why or why not?

6.6 You observe a quasar at a redshift $z = 5.0$, and determine that the observed flux of light from the quasar varies on a timescale $\delta t_0 = 3$ days. If the observed variation in flux is due to a variation in the intrinsic luminosity of the quasar, what was the variation timescale δt_e at the time the light was emitted? For the light from the quasar to vary on a timescale δt_e, the bulk of the light must come from a region of physical size $R \leq R_{\max} = c(\delta t_e)$. What is $R_{\max}$ for the observed quasar? What is the angular size of $R_{\max}$ in the Benchmark Model?