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5

Model Universes

In a spatially homogeneous and isotropic universe, the relation between the energy density $\varepsilon(t)$, the pressure $P(t)$, and the scale factor $a(t)$ is given by the Friedmann equation,

FRIEDMANN

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}, \tag{5.1}$$

the fluid equation,

(OR CONTINUITY EQUATION)

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0, \tag{5.2}$$

and the equation of state,

EOS (OR ACCELLERATION EQ)

$$\boxed{P = w\varepsilon.} \tag{5.3}$$

In principle, given the appropriate boundary conditions, we can solve Equations 5.1, 5.2, and 5.3 to yield $\varepsilon(t)$, $P(t)$, and $a(t)$ for all times, past and future. In reality, the evolution of our universe is complicated by the fact that it contains different components with different equations of state. Let's start by seeing how the energy density ε of the different components changes as the universe expands.

5.1 Evolution of Energy Density

The universe contains nonrelativistic matter and radiation – that's a conclusion as firm as the earth under your feet and as plain as daylight. Thus, the universe contains components with both $w = 0$ and $w = 1/3$. It contains dark energy that is consistent with being a cosmological constant ($w = -1$). Moreover, the possibility exists that it may contain still more exotic components, with different values of w. Fortunately for the cause of simplicity, the energy density and pressure for the different components of the universe are additive. Suppose that the universe contains N different components, with the ith component having an energy density ε_i

and an equation-of-state parameter w_i. We may then write the total energy density ε as the sum of the energy density of the different components:

GENERALIZE (4.55) $P = w \cdot \varepsilon$ $\forall$ COMPONENTS i

$$\boxed{\varepsilon = \sum_i \varepsilon_i.} \tag{5.4}$$

The total pressure P is the sum of the pressures of the different components:

$$P = \sum_i w_i \varepsilon_i. = \sum_i P_i \tag{5.5}$$

Because the energy densities and pressures add in this way, the fluid equation must hold for each component separately, as long as there is no interaction between the different components. If this is so, then the component with equation-of-state parameter w_i obeys the equation

GENERALIZED FLUID EQUATION (4.44), (4.52), (4.67) $\Rightarrow$

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(\varepsilon_i + P_i) = 0 \tag{5.6}$$

or

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(1 + w_i)\varepsilon_i = 0 \Rightarrow \dot{\varepsilon}_i = \frac{-3\dot{a}}{a}(1+w_i)\varepsilon_i \tag{5.7}$$

Equation 5.7 can be rearranged to yield

$$\frac{d\varepsilon_i}{\varepsilon_i} = -3(1 + w_i)\frac{da}{a}. \tag{5.8}$$

$\frac{d\varepsilon_i}{da}$

H/W 5.1 * Show (5.9)

If we assume that w_i is constant, then

* What w-component dominates for:
1) $a \to 0$ ($z \to \infty$)
2) $a \to \infty$ ($t \to \infty$)

$$\boxed{\varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}.} \tag{5.9}$$

* Discuss the cases $w=0$ & $w=\frac{1}{3}$

Note that Equation 5.9 is derived solely from the fluid equation and the equation of state; the Friedmann equation doesn't enter into it.

From Equation 5.9, we conclude that the energy density ε_m associated with nonrelativistic matter decreases as the universe expands with the dependence

($w=0$) MATTER DOMINATED
$$\varepsilon_m(a) = \varepsilon_{m,0}/a^3. \quad \text{(Volume Law)} \Rightarrow \tag{5.10}$$
$w=0$

The energy density in radiation, ε_r, drops at the steeper rate

($w=\frac{1}{3}$) RADIATION DOMINATED
$$\varepsilon_r(a) = \varepsilon_{r,0}/a^4. \quad \text{(Cosmic Stephan Boltzmann Law)} \Rightarrow \tag{5.11}$$
$w=\frac{1}{3}$

Why this difference between matter and radiation? We may write the energy density of either component in the form $\varepsilon = nE$, where n is the number density of particles and E is the mean energy per particle. For both relativistic and nonrelativistic particles, the number density has the dependence $n \propto a^{-3}$ as the universe expands, assuming that particles are neither created nor destroyed.

The energy of nonrelativistic particles, shown in the top panel of Figure 5.1, is contributed solely by their rest mass ($E = mc^2$) and remains constant as the universe expands. Thus, for nonrelativistic matter, $\varepsilon_m = nE = n(mc^2) \propto a^{-3}$.

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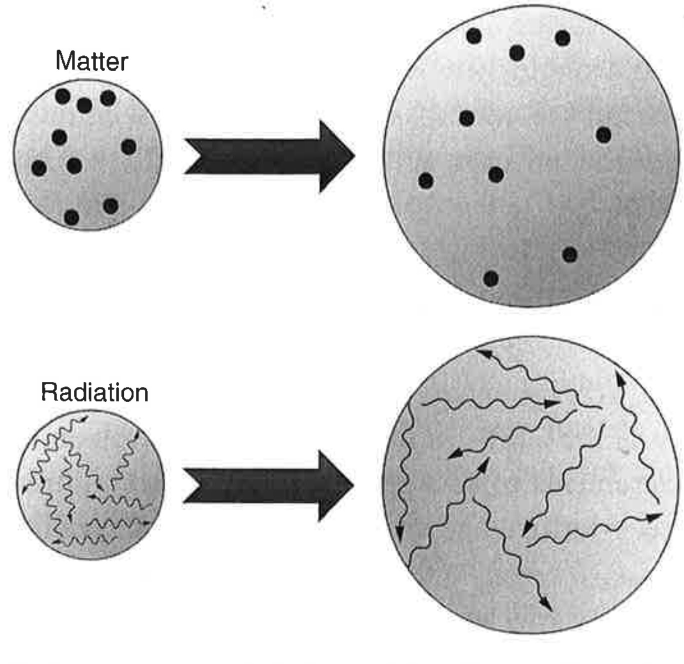


$\varepsilon_m \propto a^{-3} = (1+z)^3$

$\varepsilon_r \propto a^{-4} = (1+z)^4$

Figure 5.1 The dilution of nonrelativistic particles ("matter") and relativistic particles ("radiation") as the universe expands.

The energy of photons or other massless particles, shown in the bottom panel of Figure 5.1, has the dependence $E = hc/\lambda \propto a^{-1}$, since their wavelength λ expands along with the expansion of the universe. Thus, for photons and other massless particles, $\varepsilon_r = nE = n(hc/\lambda) \propto a^{-3}a^{-1} \propto a^{-4}$. $\Leftrightarrow$ COSMIC STEPHAN-BOLTZMANN

Although we've explained why photons have an energy density $\varepsilon_r \propto a^{-4}$, the explanation required the assumption that photons are neither created nor destroyed. This assumption is wrong: photons are always being created by luminous objects and absorbed by opaque objects.[1] However, it turns out that the energy density of the cosmic microwave background is larger than the energy density of all the photons emitted by all the stars in the history of the universe. To see why this is true, remember, from Section 2.5, that the present energy density of the CMB, which has a temperature $T_0 = 2.7255\,\mathrm{K}$, is

(2.29) + (2.33), (2.34)

$$\varepsilon_{\mathrm{CMB},0} = \alpha T_0^4 = 4.175 \times 10^{-14}\,\mathrm{J\,m^{-3}} = 0.2606\,\mathrm{MeV\,m^{-3}}. \tag{5.12}$$

Expressed as a fraction of the critical density, the CMB has a density parameter

(4.31)

$$\Omega_{\mathrm{CMB},0} = \frac{\varepsilon_{\mathrm{CMB},0}}{\varepsilon_{c,0}} = \frac{0.2606\,\mathrm{MeV\,m^{-3}}}{4870\,\mathrm{MeV\,m^{-3}}} = 5.35 \times 10^{-5}. \tag{5.13}$$

Although the energy density of the CMB is small compared to the critical density, it is large compared to the energy density of starlight. Galaxy surveys tell us that the present luminosity density of galaxies is

$$\Psi \approx 1.7 \times 10^8\,\mathrm{L_\odot\,Mpc^{-3}} \approx 2.2 \times 10^{-33}\,\mathrm{watts\,m^{-3}}. \tag{5.14}$$

[1] The Sun, for instance, is emitting 10^{45} photons every second, and thus acts as a glaring example of photon non-conservation.

(By terrestrial standards, the universe is not a well-lit place; this luminosity density is equivalent to a single 30-watt bulb within a sphere 1 AU in radius.) As a *very* rough estimate, let's assume that galaxies have been emitting light at this rate for the entire age of the universe, $t_0 \approx H_0^{-1} \approx 4.5 \times 10^{17}$ s. This gives an energy density in starlight of

Ch 2 (between Ryden ps 20 & 21)

$$\varepsilon_{\text{starlight},0} \sim \Psi t_0 \sim (2.2 \times 10^{-33}\,\text{J}\,\text{s}^{-1}\,\text{m}^{-3})(4.5 \times 10^{17}\,\text{s})$$
$$\sim 10^{-15}\,\text{J}\,\text{m}^{-3} \sim 0.006\,\text{MeV}\,\text{m}^{-3} \Leftrightarrow 10\,\text{nW}\,\text{m}^{-2}\,\text{sr}^{-1} \text{ observed} \qquad (5.15)$$

Thus, we expect the average energy density of starlight to be just a few percent of the energy density of the CMB. In fact, the estimate given above is a very rough one indeed. Measurements of background radiation from ultraviolet to infrared, including both direct starlight and starlight absorbed and reradiated by dust, yield the larger value $\varepsilon_{\text{starlight}}/\varepsilon_{\text{CMB}} \approx 0.1$. In the past, however, the ratio of starlight density to CMB density was smaller than it is today. For most purposes, it is an acceptable approximation to ignore non-CMB photons when computing the mean energy density of photons in the universe.

The cosmic microwave background, remember, is a relic of the time when the universe was hot and dense enough to be opaque to photons. If we extrapolate further back, we reach a time when the universe was hot and dense enough to be opaque to neutrinos. As a consequence, there should be a cosmic *neutrino* background today, analogous to the cosmic microwave background. The energy density in neutrinos should be comparable to, but not exactly equal to, the energy density in photons. A detailed calculation indicates that the energy density of each neutrino flavor should be

$$\varepsilon = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \varepsilon_{\text{CMB}} = 0.227\,\varepsilon_{\text{CMB}} = \varepsilon_\nu \qquad (5.16)$$

(The above result assumes that the neutrinos are relativistic, or, equivalently, that their energy is much greater than their rest energy $m_\nu c^2$.) The density parameter of the cosmic neutrino background, taking into account all three flavors of neutrino,
(5.16)×3 ⇒ should then be $\Omega_\nu = 0.681\Omega_{\text{CMB}}$, as long as all neutrino flavors are relativistic. The mean energy per neutrino will be comparable to, but not exactly equal to, the mean energy per photon:

(2.36) $E^{\text{CMB}}_{\text{mean}} \simeq 6.34 \times 10^{-4}\,\text{eV}$ $\Rightarrow 3\varepsilon_\nu = 0.68\,\varepsilon_{\text{CMB}}$

$$E_\nu \approx \frac{\cancel{5}\ 4.36 \times 10^{-4}\,\text{eV}}{a}, \qquad (5.17)$$

as long as $E_\nu > m_\nu c^2$. When the mean energy of a particular neutrino species drops to $\sim m_\nu c^2$, then it makes the transition from being "radiation" to being "matter."

If all neutrino species were effectively massless today, with $m_\nu c^2 \ll 5 \times 10^{-4}$ eV, then the present density parameter in radiation would be

$$\Omega_{r,0} = \Omega_{\text{CMB},0} + \Omega_{\nu,0} = \underbrace{5.35 \times 10^{-5}}_{(5.15)} + \underbrace{3.65 \times 10^{-5}}_{(5.16)\times 3} = 9.00 \times 10^{-5}. \quad \text{SMALL !} \qquad (5.18)$$

We know the energy density of the cosmic microwave background with high precision. We can calculate theoretically what the energy density of the cosmic neutrino background should be. The total energy density of nonrelativistic matter, and that of dark energy, is not quite as well determined. The available evidence favors a universe in which the density parameter for matter is currently $\Omega_{m,0} \approx 0.31$, while the density parameter for the cosmological constant is currently $\Omega_{\Lambda,0} \approx 0.69$. Thus, when we want to employ a model that matches the observed properties of the real universe, we will use what I call the "Benchmark Model"; this model has $\Omega_{r,0} = 9.0 \times 10^{-5}$ in radiation, $\Omega_{m,0} = 0.31$ in nonrelativistic matter, and $\Omega_{\Lambda,0} = 1 - \Omega_{r,0} - \Omega_{m,0} \approx 0.69$ in a cosmological constant.[2]

2018 Planck $\Omega_{m,0} \simeq 0.32$ $\Omega_{\Lambda,0} \simeq 0.68$ "CONCORDANCE MODEL"

In the Benchmark Model, at the present moment, the ratio of the energy density in Λ to the energy density in matter is

$$\frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \approx \frac{0.69}{0.31} \approx 2.23. \qquad \frac{0.68}{0.32} \simeq 2.13 \tag{5.19}$$

In the language of cosmologists, the cosmological constant is "dominant" over matter today in the Benchmark Model. In the past, however, when the scale factor was smaller, the ratio of densities was

CONSTANT !!

$$\boxed{\frac{\varepsilon_\Lambda(a)}{\varepsilon_m(a)} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}/a^3} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} a^3} = 1 \Leftrightarrow (5.21) \tag{5.20}$$

If the universe has been expanding from an initial very dense state, at some moment in the past, the energy density of matter and Λ must have been equal. This moment of matter–Λ equality occurred when the scale factor was

H/W 5.2

* For what value of $z_{\Lambda m}$ is $\varepsilon_\Lambda = \varepsilon_m$

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \approx \left(\frac{0.31}{0.69}\right)^{1/3} \approx \cancel{0.766}. \; 0.778 = \frac{1}{1.286} \tag{5.21}$$

* Discuss in terms of merger rate(z) LF(z), SFR(z)

Similarly, the ratio of the energy density in matter to the energy density in radiation is currently

(5.19) (5.18) $\Rightarrow$

$$\frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} = \frac{\Omega_{m,0}}{\Omega_{r,0}} \approx \frac{0.32}{9.0 \times 10^{-5}} \approx \cancel{3400}, \; 3500 = \frac{1}{a_{rm}} = 1 + z_{rm} \tag{5.22}$$

if all three neutrino flavors in the cosmic neutrino background are assumed to be relativistic today. (It's even larger if some or all of the neutrino flavors are massive enough to be nonrelativistic today.) Thus, matter is now strongly dominant over radiation. However, in the past, the ratio of matter density to energy density was

$$\frac{\varepsilon_m(a)}{\varepsilon_r(a)} = \frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} a. = \frac{\varepsilon_{m,0} \times a^{-3}}{\varepsilon_{r,0} \times a^{-4}} \tag{5.23}$$

Thus, the moment of radiation–matter equality took place when the scale factor was

[2] Note that the Benchmark Model is defined to be spatially flat.

$$a_{rm} = \frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} \approx \frac{1}{3400} \approx 2.9 \times 10^{-4} \Longrightarrow z_{M\text{-}Rad\ EQUAL} \simeq \cancel{3400}\ 3500 \quad (5.24)$$

Note that as long as a neutrino's mass is $m_\nu c^2 \ll (3400)(5 \times 10^{-4}\,\text{eV}) \sim 2\,\text{eV}$, then it was relativistic at a scale factor $a = 1/3400$, and hence would have been "radiation" then even if it is "matter" today.

Positive

To generalize, if the universe contains different components with different values of w, Equation 5.9 tells us that in the limit $a \to 0$, the component with the largest positive value of w is dominant. If the universe expands forever, then as $a \to \infty$, the component with the smallest value of w is dominant. The evidence indicates we live in a universe where radiation ($w = \frac{1}{3}$) was dominant during the early stages, followed by a period when matter ($w = 0$) was dominant, followed by a period when the cosmological constant ($w = -1$) is dominant.

In a continuously expanding universe, the scale factor a is a monotonically increasing function of t. Thus, in a continuously expanding universe, the scale factor a can be used as a surrogate for the cosmic time t. We can refer, for instance, to the moment when $a = 0.766$ with the assurance that we are referring to a unique moment in the history of the universe. In addition, because of the simple relation between scale factor and redshift, $1+z = 1/a$, cosmologists often use redshift as a surrogate for time. For example, they make statements such as, "Matter–lambda equality took place at a redshift $z_{m\Lambda} \approx 0.31$." That is, light that was emitted at the time of matter–lambda equality is observed by us with its wavelength stretched by a factor $1 + z_{m\Lambda} \approx 1.31$. ($z_{m\Lambda} \simeq 0.286 - 0.296$ for $\Omega_m = 0.32 - 0.315$)

One reason why cosmologists use scale factor or redshift as a surrogate for time is that the conversion from a to t is not simple to calculate in a multiple-component universe like our own. In a universe with many components, the Friedmann equation can be written in the form

FRIEDMANN EQ $\forall\, w$!:

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_i \varepsilon_{i,0} \overbrace{a^{-1-3w_i}}^{a^{-(1+3w_i)}} - \frac{\kappa c^2}{R_0^2}. \quad \Leftarrow (5.1)+(5.9) \quad (5.25)$$

$w=1/3$ a^{-2} Rad
$w=0$ a^{-1} Matter
$w=-\frac{1}{3}$ a^0 Curv
$w=-1$ a^2 Λ

Each term on the right-hand side of Equation 5.25 has a different dependence on scale factor; radiation ($w=1/3$) contributes a term $\propto a^{-2}$, matter (non-rel) contributes a term $\propto a^{-1}$ ($w=0$), curvature contributes a term independent of a, and the cosmological constant Λ contributes a term $\propto a^2$. Solving Equation 5.25 for a multiple-component model like the Benchmark Model does not yield a simple analytic form for $a(t)$. However, looking at single-component universes, in which there is only one term on the right-hand side of Equation 5.25, yields useful insight into the physics of an expanding universe.

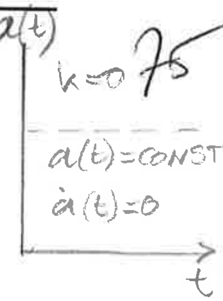
5.2 Empty Universes

A particularly simple universe is one that is empty – no radiation, no matter, no cosmological constant, no contribution to ε of any sort. For this universe, the

ALLOWED SOL: K=0 ⇒ MINKOWSKI (3.37) SPACE IS FLAT, STATIC

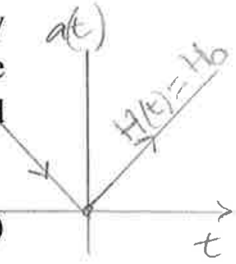
K=−1 (MILNE MODEL)

$$\dot{a}^2 = -\frac{\kappa c^2}{R_0^2} \cdot \Leftrightarrow \frac{dr}{dt} = \pm c \quad (\text{see } 3.40) \tag{5.26}$$



One solution to this equation has $\dot{a} = 0$ and $\kappa = 0$. An empty, static, spatially flat universe is a permissible solution to the Friedmann equation. This is the universe whose geometry is described by the Minkowski metric of Equation 3.37, and in which all the transformations of special relativity hold true.

However, Equation 5.26 tells us that it is also possible to have an empty universe with $\kappa = -1$. (Positively curved empty universes are forbidden, since that would require an imaginary value of $\dot{a}$ in Equation 5.26.) A negatively curved empty universe must be expanding or contracting, with



EMPTY UNIVERSE ⇒ $\varepsilon_i \equiv 0$
NEGATIVE CURVATURE ⇒ K=−1
FRIEDMANN EQ $\forall \varepsilon_i$ (5.25) ✓ ⇒

$$\dot{a} = \pm \frac{c}{R_0} \equiv \pm H_0 = \pm \frac{1}{t_0} \tag{5.27}$$

In an expanding empty universe, integration of this relation yields a scale factor of the form[3]

$$a(t) = \frac{t}{t_0}, \text{ for } \boxed{t_0 = \frac{R_0}{c} \equiv H_0^{-1}} \tag{5.28}$$

where $t_0 = R_0/c$. In Newtonian terms, if there's no gravitational force at work, then the relative velocity of any two points is constant, and the scale factor a simply increases linearly with time in an empty universe. ⇒ $H(t) = H_0 =$ CONSTANT FOR ALL t!

The scale factor in an empty, expanding universe is shown as the dashed line in Figure 5.2. Note that in an empty universe, $t_0 = H_0^{-1}$; with nothing to speed or slow the expansion, the age of the universe is exactly equal to the Hubble time. (NO ACCELL OR DECELL!)

An empty, expanding universe might seem nothing more than a mathematical curiosity.[4] However, if a universe has a density ε that is very small compared to the critical density ε_c (that is, if $\Omega \ll 1$), then the linear scale factor of Equation 5.28 is a good approximation to the true scale factor. Imagine you are in an expanding universe with a negligibly small value for the density parameter Ω, so that you can reasonably approximate it as an empty, negatively curved universe, with $t_0 = H_0^{-1} = R_0/c$. You observe a distant light source, such as a galaxy, which has a redshift z. The light you observe now, at $t = t_0$, was emitted at an earlier time, $t = t_e$. In an empty expanding universe,

(3.60):

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)} = \frac{\lambda_0}{1} \Longrightarrow \frac{\lambda_0}{\lambda_e} \equiv 1 + z = \frac{1}{a(t_e)} \overset{(5.28)}{=} \frac{t_0}{t_e}, \Rightarrow t_e = t_0 \cdot a(t_e) \tag{5.29}$$

so it is easy to compute the time when the light you observe from the source was emitted:

OUR FIRST AND SIMPLEST DIRECT RELATION BETWEEN COSMIC TIME AND REDSHIFT!

$$\boxed{t_e \overset{(5.29)}{=} \frac{t_0}{1+z} \overset{(5.28)}{=} \frac{H_0^{-1}}{1+z}.} \tag{5.30}$$

$t_0 \equiv H_0^{-1}$ FOR MILNE MODEL

[3] Such an empty, negatively curved, expanding universe is sometimes called a *Milne universe*, after the cosmologist E. A. Milne, who pioneered its study in the 1930s.

[4] If a universe contains nothing, there will be no observers in it to detect the expansion.

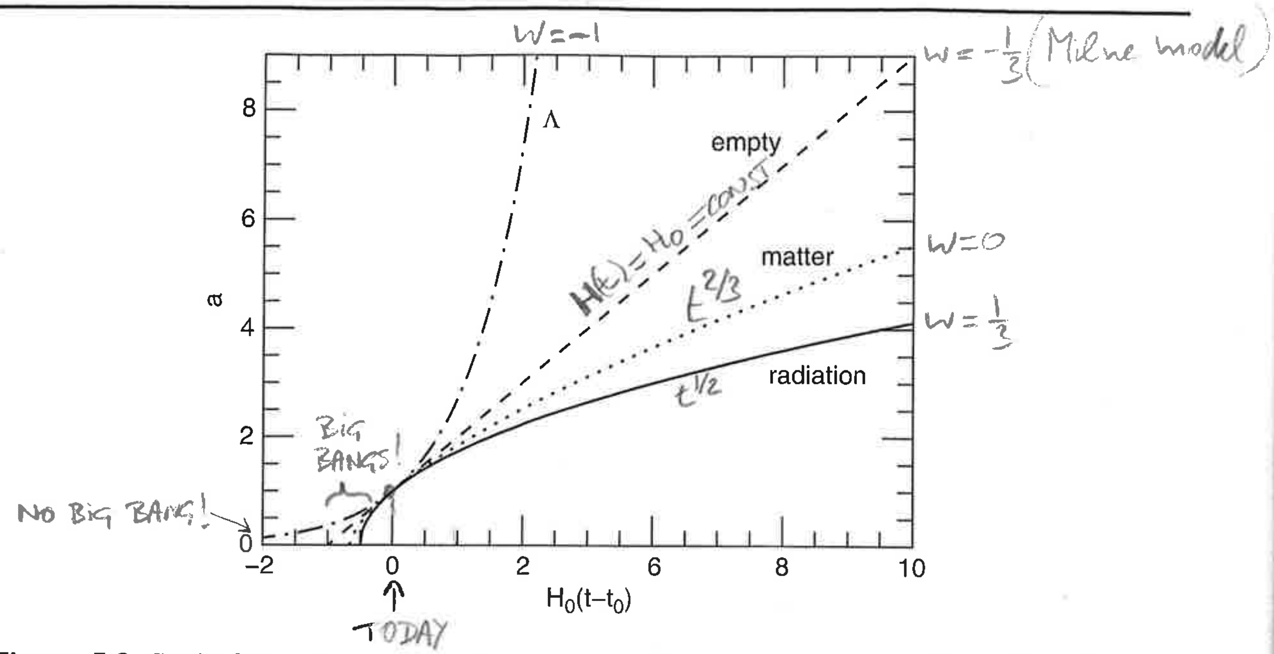


Figure 5.2 Scale factor versus time for an expanding, empty universe (dashed), a flat, matter-dominated universe (dotted), a flat, radiation-dominated universe (solid), and a flat, Λ-dominated universe (dot-dash).

When observing a galaxy with a redshift z, in addition to asking, "When was the light from that galaxy emitted?" you may also ask, "How far away is that galaxy?" In Section 3.5 we saw that in any universe described by a Robertson–Walker metric, the current proper distance from an observer at the origin to a galaxy at coordinate location (r, θ, ϕ) is (see Equation 3.44)

(3.44) at t_0: $d_p(t) \overset{\downarrow}{=} a(t)\int_0^r dr = a(t)\cdot r$ $\boxed{d_p(t_0) \equiv r}$

PROPER DISTANCE $\rightarrow$ $$d_p(t_0) = a(t_0)\int_0^r dr = r. \leftarrow \text{COMOVING RADIAL DISTANCE} \qquad (5.31)$$

since $a(t_0) \equiv 1$

Moreover, if light is emitted by the galaxy at time t_e and detected by the observer at time t_0, the null geodesic followed by the light satisfies Equation 3.54:

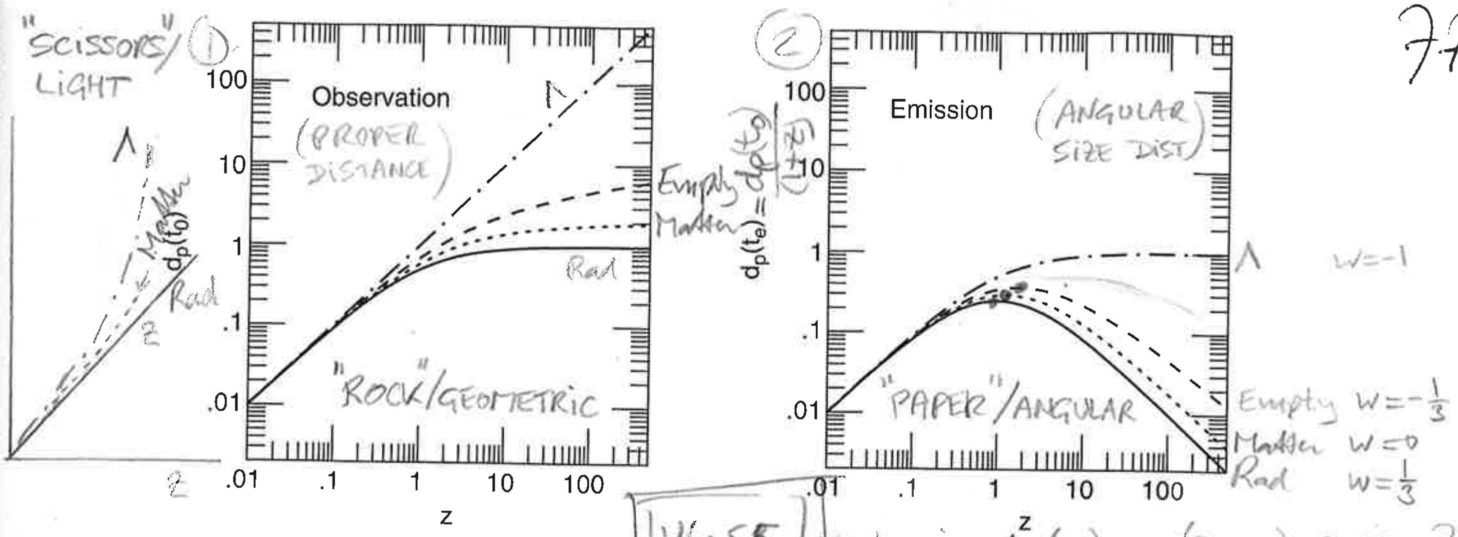
(3.54) $\Longrightarrow$ $$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r. \overset{(5.31)}{\equiv} d_p(t_0) \qquad (5.32)$$

Thus, the current proper distance from you (the observer) to the galaxy (the light source) is

PROPER DISTANCE IN ALL FRIEDMANN-RW UNIVERSES:
$$\boxed{d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)}.} \qquad (5.33)$$

Equation 5.33 holds true in any universe whose geometry is described by a Robertson–Walker metric. In the specific case of an empty expanding universe (Milne) (5.29) $\Longrightarrow$ $\boxed{a(t) = t/t_0}$, and thus

$\frac{d\ln f}{dx} = \frac{1}{f}\frac{df}{dx} \Rightarrow \int \frac{1}{f}df = \ln f \Longrightarrow$ $$d_p(t_0) = ct_0\int_{t_e}^{t_0}\frac{dt}{t} = ct_0 \ln\left(\frac{t_0}{t_e}\right) = \overset{R_0}{\overset{\downarrow}{\frac{c}{H_0}}}\ln\left(\frac{1}{a(t)}\right) \qquad (5.34)$$



$\boxed{H/w\ 5.5}$ * what is $d_p(t_e)_{max}(z_{max})$ for $\forall w$?

Figure 5.3 The proper distance to an object with observed redshift z, measured in units of the Hubble distance, c/H_0. Left panel: the proper distance at the time the light is observed. Right panel: proper distance at the time the light was emitted. Line types are the same as those of Figure 5.2.

(1) PROPER DISTANCE: (1) Expressed in terms of the redshift z of the observed galaxy,

PROPER DISTANCE AT TIME OF OBSERVATION:

$$d_p(t_0) = \frac{c}{H_0}\ln(1+z) = R_0 \ln(1+z) \quad \simeq \frac{c}{H_0}\cdot z \text{ for } z \ll 1, \quad \simeq \frac{c}{H_0}\cdot \ln(z) \text{ for } z \gg 1 \tag{5.35}$$

This relation is plotted as the dashed line in the left panel of Figure 5.3. In the limit $z \ll 1$, there is a linear relation between d_p and z. In the limit $z \gg 1$, however, $d_p \propto \ln z$ in an empty expanding universe.

In an empty expanding universe, we can see objects that are currently at an arbitrarily large distance. At first glance, it may seem counterintuitive that we can see a light source at a proper distance much greater than c/H_0 when the age of the universe is only $1/H_0$. However, remember that $d_p(t_0)$ is the proper distance to the light source at the time of observation; at the time of *emission*, the proper distance $d_p(t_e)$ was smaller by a factor $a(t_e)/a(t_0) = 1/(1+z)$. In an empty expanding universe, the proper distance at the time of emission was (2)

(2) PROPER DISTANCE AT TIME OF EMISSION

$$d_p(t_e) = \frac{c}{H_0}\frac{\ln(1+z)}{(1+z)}, \quad \left(= \frac{\text{ANGULAR SIZE}}{\text{DISTANCE}}\right) \tag{5.36}$$

$\boxed{d_p(t_e) = \frac{d_p(t_0)}{(1+z)}}$

shown as the dashed line in the right panel of Figure 5.3. In an empty expanding universe, $d_p(t_e)$ has a maximum for objects with a redshift $z = e - 1 \approx 1.72$, where $d_p(t_e) = (1/e)\,c/H_0 \approx 0.37\,c/H_0$. Objects with much higher redshifts are seen as they were very early in the history of the universe, when their proper distance from the observer was very small.

OBEYS $\theta = \frac{l}{d_{p_e}(z)}$ AT ALL z (SMALL θ APPROX)

(3) LUMINOSITY DISTANCE $d_L(z) = d_p(t_0) \times (1+z) \Rightarrow$ (3) $\boxed{d_L(z) = \frac{c}{H_0}(1+z)\ln(1+z)}$

OBEYS INVERSE SQUARE LAW AT $\forall z$! $\simeq \frac{c}{H_0} z$ for $z \ll 1$, $\simeq \frac{c}{H_0} z \ln(z)$ for $z \gg 1$

5.3 Single-component Universes

Setting the energy density ε equal to zero is one way of simplifying the Friedmann equation. Another way is to set $\kappa = 0$ and to demand that the universe contain

From Longair's Book, Galaxy Formation

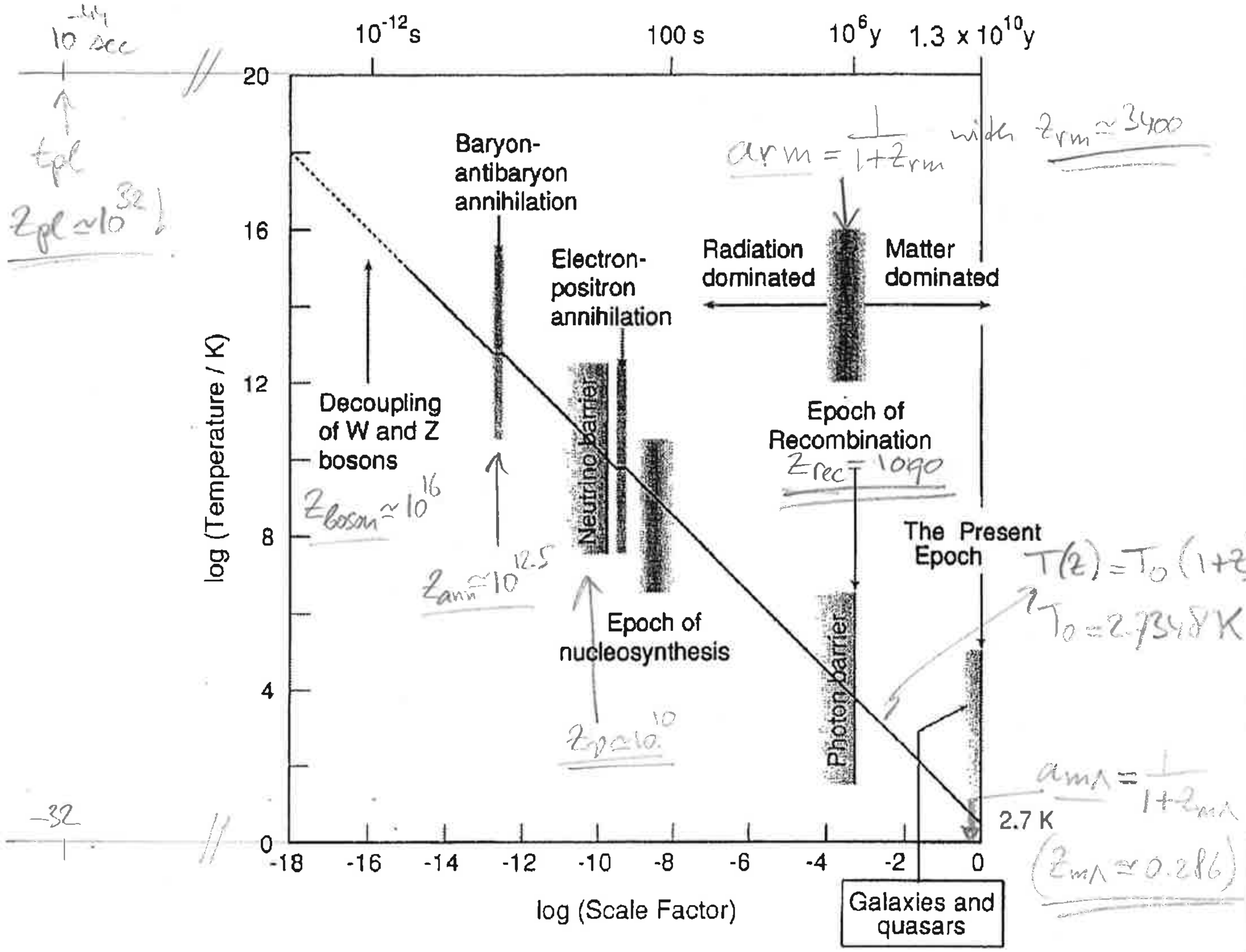


Fig. 9.3. The thermal history of the Cosmic Microwave Background Radiation according to the standard Big Bang picture. The radiation temperature decreases as $T_r \propto a^{-1}$ except for small discontinuities as different particle–antiparticle pairs annihilate at $kT \approx mc^2$. Various important epochs in the standard picture are indicated, including the neutrino and photon barriers. In the standard model, the Universe is optically thick to neutrinos and photons prior to these epochs. An approximate *time-scale* is indicated along the *top* of the diagram

energies $h\nu \geq 13.6\,\mathrm{eV}$ in the high energy region of the Planck distribution to ionise all the neutral hydrogen present in the intergalactic medium. We will often refer to the region of the Planck spectrum with photon energies $E = h\nu \gg kT$ as the *Wien region* It may at first appear strange that the temperature is not closer to 150,000 K, the temperature at which $\langle h\nu \rangle = kT = 13.6\,\mathrm{eV}$, the ionisation potential of neutral hydrogen. The important points to note are that the photons far outnumber the baryons in the intergalactic medium and that there is a broad range of photon energies present in the Planck distribution.

It is a useful calculation to work out the fraction of photons in the Wien region of the Planck distribution with energies $h\nu \geq E$ in the limit $h\nu \gg kT$. Their number only a single component, with a single value of w. In such a spatially flat, single-component universe, the Friedmann equation takes the form

(5.25) GENERALIZED FRW FOR $K=0$: (FLAT)

$$\dot{a}^2 = \frac{8\pi G\varepsilon_0}{3c^2}a^{-(1+3w)}. \tag{5.37}$$

To solve this equation, we first make the educated guess that the scale factor has the power-law form $\boxed{a \propto t^q}$. (5.37a) The left-hand side of Equation 5.37 is then $\propto t^{2q-2}$, and the right-hand side is $\propto t^{-(1+3w)q}$, yielding the solution

HW 5.3
* SHOW 5.38, 5.39, 5.41, 5.42
* & DISCUSS ITS MEANING
* CALCULATE t_0 for CURRENT ε_0 if $W=0$

$$\boxed{q = \frac{2}{3+3w},} \quad \text{FOR } w \neq -1 \tag{5.38}$$

with the restriction $w \neq -1$. With the proper normalization, the scale factor in a spatially flat, single-component universe is

$$\frac{1}{(1+z(t))} \equiv a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)} \simeq \begin{cases} \left(\frac{t}{t_0}\right)^{1/2} & \text{for } W=\frac{1}{3} \text{ Rad} \\ \left(\frac{t}{t_0}\right)^{2/3} & \text{for } W=0 \text{ NON-Rel Matter} \end{cases} \tag{5.39}$$

EdS

The age of the universe, t_0, is linked to the present energy density by the relation

$$t_0 = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\varepsilon_0}\right)^{1/2} \Longleftrightarrow \varepsilon_0 = \frac{c^2}{6\pi G(1+w)^2}t_0^{-2} \tag{5.40}$$

(5.45)

The Hubble constant in such a universe is

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+w)}t_0^{-1}. \tag{5.41}$$

The age of the universe, in terms of the Hubble time, is then

$$\boxed{t_0 = \frac{2}{3(1+w)}H_0^{-1}.} \begin{cases} \frac{1}{2}H_0^{-1} & \text{FOR } W=\frac{1}{3} \text{ Rad} \\ \frac{2}{3}H_0^{-1} & \text{FOR } W=0 \text{ MATTER} \end{cases} \tag{5.42}$$

In a spatially flat universe, if $w > -1/3$, the universe is *younger* than the Hubble time. If $w < -1/3$, the universe is *older* than the Hubble time.

As a function of scale factor, the energy density of a component with equation-of-state parameter w is (5.9) + (5.39)

$$\varepsilon(a) = \varepsilon_0 a^{-3(1+w)} \quad \varepsilon_0\left(t^{\frac{2}{3(1+w)}}\right)^{-3(1+w)} = \varepsilon_0 t^{-2} \tag{5.43}$$

$K=0$

so in a spatially flat universe with only a single component, the energy density as a function of time is (combining Equations 5.39 and 5.43)

$$\boxed{\varepsilon(t) = \varepsilon_0\left(\frac{t}{t_0}\right)^{-2},} \tag{5.44}$$

!

regardless of the value of w. Making the substitution

(4.30)

$$\varepsilon_0 = \varepsilon_{c,0} = \frac{3c^2}{8\pi G}H_0^2 = \frac{c^2}{6\pi G(1+w)^2}t_0^{-2}, \quad \varepsilon_0 = \frac{c^2}{6\pi G(1+w)^2}t_0^{-2} \tag{5.45}$$

USING (1.1)–(1.4): $\left.\begin{array}{l} E_P/\ell_P^3 = \hbar^{-1} c^7 G^{-2} \\ t_P^2 = G\hbar c^{-5} \end{array}\right\} \Rightarrow \dfrac{E_P}{\ell_P^3 t_P^{-2}} = \dfrac{c^2}{G}$!

Equation 5.44 can be written in the form

THIS VERSION EVEN HOLDS FOR $t \gtrsim t_P$!

$$\varepsilon(t) = \frac{1}{6\pi(1+w)^2}\frac{c^2}{G}t^{-2} = \frac{1}{6\pi(1+w)^2}\frac{E_P}{\ell_P^3}\left(\frac{t}{t_P}\right)^{-2}. \tag{5.46}$$

Suppose yourself to be in a spatially flat, single-component universe. If you see a galaxy, or other distant light source, with a redshift z, you can use the relation

$\left.\begin{array}{l} 1/(5.39) \\ +(5.29) \end{array}\right\} \Rightarrow$

$$1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)} \quad \text{with } a(t_0) \equiv 1 \tag{5.47}$$

to compute the time t_e at which the light from the distant galaxy was emitted:

(5.42) $t_0 = \frac{2}{3(1+w)} H_0^{-1} \Rightarrow$

$$t_e = \frac{t_0}{(1+z)^{3(1+w)/2}} = \boxed{\left(\frac{2}{3(1+w)H_0}\right)\left(\frac{1}{(1+z)^{3(1+w)/2}}\right) = t_e(z)} \tag{5.48}$$

OUR FIRST $t(z)$ RELATION. VALID $\forall w$! [COSMIC CLOCK(z)]

The current proper distance to the galaxy is t_0 in (5.42)

HW 5.4

* SHOW 5.48–5.50

* PLOT & DISCUSS $d_p(z), t(z)$ FOR $w = 0, \frac{1}{3}$

$$d_p(t_0) \overset{(5.33)}{=} c\int_{t_e}^{t_0}\frac{dt}{a(t)} \overset{(5.48)}{=} ct_0\frac{3(1+w)}{1+3w}\left[1 - (t_e/t_0)^{(1+3w)/(3+3w)}\right] = d_p(t_0) \tag{5.49}$$

(5.39) or (5.47)

when $w \neq -1/3$. In terms of H_0 and z rather than t_0 and t_e, the current proper distance is

PROPER DISTANCE:

$$\boxed{d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1 - (1+z)^{-(1+3w)/2}\right] = d_p(z)} \tag{5.50}$$

OUR FIRST $d_p(z)$ RELATION! VALID $\forall w$!

The most distant object you can see (in theory) is one for which the light emitted at $t = 0$ is just now reaching us at $t = t_0$. The proper distance (at the time of observation) to such an object is called the *horizon distance*.[5] Here on Earth, the horizon is a circle centered on you, beyond which you cannot see because of the Earth's curvature. In the universe, the horizon is a spherical surface centered on you, beyond which you cannot see because light from more distant objects has not had time to reach you. In a universe described by a Robertson–Walker metric, the current horizon distance is

$$d_{\rm hor}(t_0) = c\int_0^{t_0}\frac{dt}{a(t)}. \tag{5.51}$$

In a spatially flat universe, the horizon distance has a finite value if $w > -1/3$. In such a case, computing the value of $d_p(t_0)$ in the limit $t_e \to 0$ (or, equivalently, $z \to \infty$) yields

$$d_{\rm hor}(t_0) \overset{(5.49)\text{ WITH } t_e=0}{=} ct_0\frac{3(1+w)}{1+3w} \overset{(5.42)}{=} \frac{c}{H_0}\left(\frac{2}{1+3w}\right) = \begin{cases} 2R_0 \text{ for } w=0 & \text{MATTER} \\ 1R_0 \text{ for } w=\frac{1}{3} & \text{RADIATION} \end{cases} \tag{5.52}$$

In a flat universe dominated by matter ($w = 0$) or by radiation ($w = 1/3$), an observer can see only a finite portion of the infinite volume of the universe.

[5] More technically, this is what's called the *particle horizon distance*; we'll continue to call it the horizon distance, for short.

The portion of the universe lying within the horizon for a particular observer is referred to as the *visible universe* for that observer. The visible universe consists of all points in space that have had sufficient time to send information, in the form of photons or other relativistic particles, to the observer. In other words, the visible universe consists of all points that are *causally connected* to the observer.

In a flat universe with $w \le -1/3$, the horizon distance is infinite, and all of space is causally connected to any observer. In such a universe with $w \le -1/3$, you could see every point in space – assuming the universe was transparent, of course. However, for extremely distant points, you would see extremely red-shifted versions of what they looked like extremely early in the history of the universe.

H/W 5.5a *Derive (5.53)–(5.57) FROM (5.36)–(5.52) FOR $w=0$ (Matter)

*SKETCH $d_p(t_0)$ and $d_p(t_e)$ vs. z using a plotting pgm

5.3.1 Matter only

Let's now look at specific examples of spatially flat universes, starting with a universe containing only nonrelativistic matter ($w = 0$).[6] The age of such a universe is

$$(5.42) \Rightarrow \qquad t_0 = \frac{2}{3H_0} = \frac{2}{3} H_0^{-1} \tag{5.53}$$

and the horizon distance is

$$(5.52) \Rightarrow \qquad d_{\rm hor}(t_0) = 3ct_0 = 2c/H_0 = 2R_0 \quad \text{with } R_0 \equiv \frac{c}{H_0} \tag{5.54}$$

The scale factor, as a function of time, is

$$(5.39) \Rightarrow \qquad a_m(t) = \left(\frac{t}{t_0}\right)^{2/3}, \quad \leftarrow \text{Einstein–de Sitter expansion (for } \Omega_m \equiv 1) \tag{5.55}$$

illustrated as the dotted line in Figure 5.2. If you see a galaxy with redshift z in a flat, matter-only universe, the proper distance to that galaxy, at the time of observation, is

$$= 2R_0\left[1 - (1+z)^{-1/2}\right] = d_p(t_0)$$

PROPER DIST

$$(5.50) \Rightarrow \quad d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{(t/t_0)^{2/3}} = 3ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] = d_p(t_0) \tag{5.56}$$

illustrated as the dotted line in the left panel of Figure 5.3. The proper distance at the time the light was emitted was smaller by a factor $1/(1+z)$:

$$(5.36) \ \text{ANGULAR SIZE DIST:} \qquad d_p(t_e) = \frac{2c}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right] = d_p(t_e) \tag{5.57}$$

illustrated as the dotted line in the right panel of Figure 5.3. In a flat, matter-only universe, $d_p(t_e)$ has a maximum for galaxies with a redshift $z = 5/4$, where $d_p(t_e) = (8/27)c/H_0 \approx 0.30c/H_0$.

H/W 5.5a

*SHOW $d_p(t_e) = \max$ for $z_{max} = 1.25$

[6] Such a universe is sometimes called an *Einstein–de Sitter universe*, after Albert Einstein and the cosmologist Willem de Sitter, who jointly wrote a paper on flat, matter-dominated universes in 1932.

H/W 5.56 * Derive (5.58)–(5.62) FROM (5.36)–(5.52) FOR $w = 1/3$ Rad
* Sketch $d_p(t_0)$ and $d_p(t_e)$ vs. z using a plotting pgm

5.3.2 Radiation only

The case of a spatially flat universe containing only *radiation* is of particular interest, since early in the history of our own universe, the radiation ($w = 1/3$) term dominated the right-hand side of the Friedmann equation (see Equation 5.25). Thus, at early times – long before the time of radiation–matter equality – the universe was well described by a spatially flat, radiation-only model. In an expanding, flat universe containing only radiation, the age of the universe is

(5.42) ⇒
$$t_0 = \frac{1}{2H_0}, = \frac{1}{2} H_0^{-1} \quad (5.58)$$

and the horizon distance at t_0 is

(5.52) ⇒
$$d_{\rm hor}(t_0) = 2ct_0 = \frac{c}{H_0} = R_0\,! \quad (5.59)$$

In the special case of a flat, radiation-only universe, the horizon distance is exactly equal to the Hubble distance, which is not generally the case. The scale factor of a flat, radiation-only universe is

(5.39) ⇒
$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}, \quad (5.60)$$
RADIATION DOMINATED HOT BB EXPANSION

illustrated as the solid line in Figure 5.2. If at a time t_0 you observe a distant light source with redshift z in a flat, radiation-only universe, the proper distance to the light source will be

PROPER DIST
(5.50) ⇒
$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{(t/t_0)^{1/2}} = 2ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/2}\right] = \boxed{\frac{c}{H_0}\left(\frac{z}{1+z}\right) = d_p(t_0) = R_0\left(\frac{z}{1+z}\right)} \quad (5.61)$$

illustrated as the solid line in the left panel of Figure 5.3. The proper distance at the time the light was emitted was

(5.36) ⇒ ANGULAR SIZE DISTANCE
$$\boxed{d_p(t_e) = \frac{c}{H_0}\frac{z}{(1+z)^2}, = d_p(t_e)} \quad (5.62)$$

illustrated as the solid line in the right panel of Figure 5.3. In a flat, radiation-dominated universe, $d_p(t_e)$ has a maximum for light sources with a redshift $z = 1$, where $d_p(t_e) = 0.25c/H_0$. ⇐ H/W 5.56 * PROVE

$w = \frac{1}{3}$

From Equation (5.46), the energy density in a flat, radiation-only universe is

(5.46), $w = 1/3$ ⇒
$$\varepsilon_r(t) = \frac{3}{32\pi}\frac{E_P}{\ell_P^3}\left(\frac{t}{t_P}\right)^{-2} \approx 0.030\frac{E_P}{\ell_P^3}\left(\frac{t}{t_P}\right)^{-2} \simeq \varepsilon_r(t) \quad (5.63) = \alpha T^4$$

Using the blackbody relation between energy density and temperature, given in Equations 2.28 and 2.29, we may assign a temperature to a universe dominated by blackbody radiation:

(2.28): $\varepsilon_\gamma = \alpha T^4$
(2.29): $\alpha = \frac{\pi^2 k^4}{15\hbar^3 c^3}$
(1.1)–(1.4)

$$T(t) = \left(\frac{45}{32\pi^2}\right)^{1/4} T_P\left(\frac{t}{t_P}\right)^{-1/2} \approx 0.61T_P\left(\frac{t}{t_P}\right)^{-1/2} \simeq \frac{1}{a(t)} \quad (5.64)$$
(2.41), (5.60)

$$\boxed{T(t) = \frac{T_0}{a(t)} = T_0(1+z)}$$

Here T_P is the Planck temperature, $T_P = 1.42 \times 10^{32}$ K. The mean energy per photon in a radiation-dominated universe is then

Fig 2.7 (2.31)

$$E_{\rm mean}(t) \approx 2.7kT(t) \approx \boxed{1.7E_P\left(\frac{t}{t_P}\right)^{-1/2} \simeq E_{\rm mean}(t)} \qquad (5.65)$$

and the number density of photons is (combining Equations 5.63 and 5.65)

$$n(t) = \frac{\varepsilon_r(t)}{E_{\rm mean}(t)} \approx \boxed{\frac{0.018}{\ell_P^3}\left(\frac{t}{t_P}\right)^{-3/2} \simeq n(t)} \qquad (5.66)$$

In a flat, radiation-only universe, as $t \to 0$, $\varepsilon_r \to \infty$ (Equation 5.63). Thus, at the instant $t = 0$, the energy density of our own universe (well approximated as a flat, radiation-only model in its early stages) was infinite, according to this analysis; this infinite energy density was provided by an infinite number density of particles (Equation 5.66), each of infinite energy (Equation 5.65). Should we take these infinities seriously? Not really, since the assumptions of general relativity, on which the Friedmann equation is based, break down at $t \approx t_P$.

Why can't general relativity be used at times earlier than the Planck time? General relativity is a classical theory; that is, it does not take into account the effects of quantum mechanics. In cosmological contexts, general relativity assumes that the energy content of the universe is smooth down to arbitrarily small scales, instead of being parceled into individual quanta. As long as a radiation-dominated universe has many quanta, or photons, within a horizon distance, then the approximation of a smooth, continuous energy density is justifiable, and we may safely use the results of general relativity. However, if there are only a few photons within the visible universe, then quantum mechanical effects *must* be taken into account, and the classical results of general relativity no longer apply. In a flat, radiation-only universe, the horizon distance grows linearly with time:

$\frac{\ell_P}{t_P} = c$ + (5.51) (5.52)

$$d_{\rm hor}(t) = 2ct = 2\ell_P\left(\frac{t}{t_P}\right), \qquad (5.67)$$

$w = 1/3$

so the volume of the visible universe at time t is

$(5.67)^3$

$$V_{\rm hor}(t) = \frac{4\pi}{3}d_{\rm hor}^3 \approx 34\ell_P^3\left(\frac{t}{t_P}\right)^3. \qquad (5.68)$$

Combining Equations 5.68 and 5.66, we find that the number of photons inside the horizon at time t is

(5.68) + (5.66)

$$N(t) = V_{\rm hor}(t)n(t) \approx 0.6\left(\frac{t}{t_P}\right)^{3/2} \simeq \left(\frac{13.8\times10^9 \cdot 3.156\times10^6}{10^{-43}\ \text{sec}}\right)^{3/2} \simeq (4.4\times10^{60})^{3/2} \simeq 9\times10^{90} = N_{ph} \qquad (5.69)$$

$N_b \simeq 10^{80}$

The quantization of the universe can no longer be ignored when $N(t) \approx 1$, equivalent to a time $t \approx 1.4t_P$.

$\frac{N_{ph}}{N_b} \simeq 10^{10}$

To accurately describe the universe at its very earliest stages, prior to the Planck time, a theory of quantum gravity is needed. Unfortunately, a complete

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theory of quantum gravity does not yet exist. Consequently, in this book, we will not deal with times earlier than the Planck time, $t \sim t_P \sim 10^{-43}$ s, when the number density of photons was $n \sim \ell_P^{-3} \sim 10^{104}\,\mathrm{m}^{-3}$, and the mean photon energy was $E_{\mathrm{mean}} \sim E_P \sim 10^{28}$ eV.

$\boxed{\text{H/W 5.5c}}$ * Show (5.70)–(5.74)
* Sketch $d_p(t_0)$ and $d_p(t_e)$ vs. z. using a plotting pgm.

5.3.3 Lambda only

$\boxed{w=-1}$

Consider a spatially flat universe in which the energy density is contributed by a cosmological constant Λ.[7] For a flat, lambda-dominated universe, the Friedmann equation takes the form

(5.37)? $w=-1$, $k=0$ FLAT $\Rightarrow$

$$\dot{a}^2 = \frac{8\pi G\varepsilon_\Lambda}{3c^2}a^2 \Leftrightarrow \left(\frac{\dot a}{a}\right)^2 = H(t)^2 = \frac{\Lambda}{3} \Leftrightarrow H(t) = \frac{\dot a}{a} = \sqrt{\frac{\Lambda}{3}} \tag{5.70}$$

(4.69)

where ε_Λ is constant with time. This equation can be rewritten in the form

(4.69) $\Lambda = \frac{8\pi G}{c^2}\varepsilon_\Lambda$ (4.29) at $t=t_0$

$$\dot{a} = H_0 a, \quad \text{with } H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_\Lambda(t_0) \tag{5.71}$$

where

$$H_0 = \left(\frac{8\pi G\varepsilon_\Lambda}{3c^2}\right)^{1/2}. \quad \text{since } \varepsilon_\Lambda(t) = \varepsilon_\Lambda(t_0) \ \forall t \tag{5.72}$$

The solution to Equation 5.71 in an expanding universe is

$$\boxed{a(t) = e^{H_0(t-t_0)} = e^{(t-t_0)/H_0^{-1}}} \tag{5.73}$$

This scale factor is shown as the dot-dashed line in Figure 5.2. A spatially flat universe with nothing but a cosmological constant is exponentially expanding; we've seen an exponentially expanding universe before, in Section 2.3, under the label "Steady State universe." In a Steady State universe, the density ε of the universe remains constant because of the continuous creation of real particles. If the cosmological constant Λ is provided by the vacuum energy, then the density ε of a lambda-dominated universe remains constant because of the continuous creation and annihilation of virtual particle–antiparticle pairs.

A flat universe containing nothing but a cosmological constant is infinitely old, and has an infinite horizon distance d_{hor}. If, in a flat, lambda-only universe, you see a light source with a redshift z, the proper distance to the light source, at the time you observe it, is Note sign-flip from (5.73), since we are integrating $\frac{1}{a(t)}$

(5.33) PROPER DIST

$$d_p(t_0) = c\int_{t_e}^{t_0} e^{H_0(t_0-t)}dt = \frac{c}{H_0}[e^{H_0(t_0-t_e)} - 1] = \frac{c}{H_0}z, = d_p(t_0) \tag{5.74}$$ $= R_0 \cdot z$

shown as the dot-dashed line in the left panel of Figure 5.3. The proper distance at the time the light was emitted was

since $(1+z) = \frac{1}{a(t)} = e^{H_0(t_0-t_e)}$

7 Such a universe is sometimes called a *de Sitter universe*, after Willem de Sitter, who pioneered its study in the year 1917.

H/W 5.5c * Add (5.74), (5.75) to the plots of H/W 5.5a, 5.5b

w=-1 ANGULAR SIZE DISTANCE

$$\boxed{d_p(t_e) = \frac{c}{H_0}\left(\frac{z}{1+z}\right) = R_0\left(\frac{z}{1+z}\right) = d_p(t_e)} \tag{5.75}$$

shown as the dot-dashed line in the right panel of Figure 5.3.

An exponentially growing universe, such as the flat lambda-dominated model, is the only universe for which $d_p(t_0)$ is linearly proportional to z for all values of z. In other universes, the relation $d_p(t_0) \propto z$ holds true only in the limit $z \ll 1$. In a flat lambda-dominated universe, a light source with $z \gg 1$ is at a distance $d_p(t_0) \gg c/H_0$ at the time of observation; however, the observed photons were emitted by the light source when it was at a distance $d_p(t_e) \approx c/H_0$. Once the light source is more than a Hubble distance from the observer, its recession velocity is greater than the speed of light, and photons from the light source can no longer reach the observer.

5.4 Multiple-component Universes

The simple models that we've examined so far – empty universes, or flat universes with a single component – continue to expand forever if they are expanding at $t = t_0$. Is it possible to have universes that stop expanding, then start to collapse? Is it possible to have universes in which the scale factor is not a simple power-law or exponential function of time? The short answer to these questions is "yes." To study universes with more complicated behavior, however, it is necessary to put aside our simple toy universes, with a single term on the right-hand side of the Friedmann equation, and look at complicated toy universes, with multiple terms on the right-hand side of the Friedmann equation.

The Friedmann equation, in general, can be written in the form

(4.26) GENERALIZED FRIEDMANN EQ:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2} = \left(\frac{\dot a}{a}\right)^2 \tag{5.76}$$

where $H \equiv \dot a/a$, and $\varepsilon(t)$ is the energy density contributed by all the components of the universe, including the cosmological constant. Equation 4.36 tells us the relation between κ, R_0, H_0, and Ω_0, (4.34)

(4.36)

$$\boxed{\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)} \Longleftrightarrow \boxed{1 - \Omega(t) = \frac{-\kappa c^2}{R_0^2 a(t)^2 H^2(t)}} \tag{5.77}$$

You can now do [H/W 4.4] !

so we can rewrite the Friedmann equation without explicitly including the curvature:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{H_0^2}{a(t)^2}(\Omega_0 - 1). \tag{5.78}$$

Dividing by H_0^2, this becomes

$$\frac{H(t)^2}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a(t)^2}, \quad \text{with } \boxed{\varepsilon_{c,0} = \frac{3c^2}{8\pi G}H_0^2} \tag{5.79}$$

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where the critical density today is

$$\varepsilon_{c,0} \equiv \frac{3c^2H_0^2}{8\pi G}. \tag{5.80}$$

We know that our universe contains matter, for which the energy density ε_m has the dependence $\varepsilon_m = \varepsilon_{m,0}/a^3$, and radiation, for which the energy density has the dependence $\varepsilon_r = \varepsilon_{r,0}/a^4$. Current evidence indicates the presence of a cosmological constant, with energy density $\varepsilon_\Lambda = \varepsilon_{\Lambda,0} =$ constant. We will therefore consider a universe with contributions from matter ($w = 0$), radiation ($w = 1/3$), and a cosmological constant ($w = -1$).[8]

In our universe, we expect the Friedmann equation to take the form

INFLATION REQUIRES:

NOTE SIGN CHANGE FROM (5.76)

$$\left.\begin{matrix}(5.4)+(5.9)\\+(5.79)\end{matrix}\right\}\Rightarrow \quad \frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}, \tag{5.81}$$

$\Omega(t) \equiv 1 \ \forall t$

= CONST

where $\Omega_{r,0} = \varepsilon_{r,0}/\varepsilon_{c,0}$, $\Omega_{m,0} = \varepsilon_{m,0}/\varepsilon_{c,0}$, $\Omega_{\Lambda,0} = \varepsilon_{\Lambda,0}/\varepsilon_{c,0}$, and $\boxed{\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} \equiv 1}$ $\Omega_{m,0} + \Omega_{\Lambda,0}$. The Benchmark Model has $\Omega_0 = 1$, and hence is spatially flat. However, although a perfectly flat universe is consistent with the data, it is not *demanded* by the data. Thus, prudence dictates that we should keep in mind the possibility that the curvature term, $(1 - \Omega_0)/a^2$ in Equation 5.81, might be nonzero.

Since $H = \dot{a}/a$, multiplying Equation 5.81 by a^2, then taking the square root, yields

$$\left(\frac{1}{H_0}\cdot\frac{da}{dt}\right) = H_0^{-1}\dot{a} = \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2 + (1 - \Omega_0)\right]^{1/2} \Rightarrow \int_0^t H_0\,dt = \int_0^a \frac{da}{[\quad]^{1/2}} \tag{5.82}$$

The cosmic time t as a function of scale factor a can then be found by performing the integral

$\boxed{\text{H/W 5.6}}$ or Team Proj

COSMIC CLOCK = COSMIC EXPANSION INTEGRAL

$$\int_0^t H_0\,dt = \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0}a^2 + (1 - \Omega_0)]^{1/2}} = H_0 t. \tag{5.83}$$

* Evaluate (5.83) for a reasonable grid of $\Omega_r, \Omega_m, \Omega_\Lambda$ values and plot $\forall\, a(t)$ [Figs 5.7] into a grid of Ω_Λ vs Ω_m [Fig 5.6]

This is not a user-friendly integral: in the general case, it doesn't have a simple analytic solution. However, for given values of $\Omega_{r,0}$, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$, it can be integrated numerically.

In many circumstances, the integral in Equation 5.83 has a simple analytic approximation to its solution. For instance, in the limit that $a \ll a_{rm} \approx 2.9\times10^{-4}$, ($z_{mr} \gtrsim 3400$) the Benchmark Model can be approximated as a flat, radiation-only universe. ($z_{m\Lambda} \lesssim 0.29$) In the limit that $a \gg a_{m\Lambda} \approx 0.77$, it can be approximated as a lambda-only universe. However, during some epochs of the universe's expansion, two of the components are of comparable density, and provide terms of roughly equal size in the Friedmann equation. During these epochs, a single-component model is a

[8] We can't rule out the possibility that the dark energy has $w \neq -1$, or the possibility that the universe contains even more exotic contributions to its energy density ($w =$?). These possible developments are left as an exercise for the reader.

poor description of the universe, and a two-component model must be utilized. For instance, at scale factors $a \sim a_{rm} \approx 2.9 \times 10^{-4}$, the Benchmark Model is approximated by a flat universe containing only radiation and matter. Such a universe is examined in Section 5.4.4. For scale factors $a \sim a_{m\Lambda} \approx 0.77$, the Benchmark Model is approximated by a flat universe containing only matter and a cosmological constant. Such a universe is examined in Section 5.4.2.

First, however, we will examine a universe that is of great historical interest to cosmology; a universe containing both matter and curvature (either negative or positive). During the mid-twentieth century, when the cosmological constant was out of fashion, cosmologists concentrated much of their interest on the study of curved, matter-dominated universes. In addition to being of historical interest, these curved, matter-dominated universes provide useful physical insight into the interplay between curvature, expansion, and density.

5.4.1 Matter + Curvature

$K=0$

Consider a universe containing nothing but pressureless matter, with $w = 0$. If such a universe is spatially flat, then it expands with time, as demonstrated in Section 5.3.1, with a scale factor

(5.55) $\Rightarrow$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}. \tag{5.84}$$

Such a flat, matter-only universe expands outward forever. Such a fate is sometimes known as the "Big Chill," since the temperature of the universe decreases monotonically with time as the universe expands. At this point, it is nearly obligatory for a cosmology text to quote T. S. Eliot: "This is the way the world ends / Not with a bang but a whimper."[9]

In a *curved* universe containing nothing but matter, the ultimate fate of the cosmos is intimately linked to the density parameter Ω_0. The Friedmann equation in a curved, matter-dominated universe (Equation 5.81) can be written in the form

(5.81) WITH MATTER + CURVATURE ONLY $\Rightarrow$

$$\frac{H(t)^2}{H_0^2} = \underbrace{\frac{\Omega_0}{a^3}}_{\text{Matter}} + \underbrace{\frac{1-\Omega_0}{a^2}}_{\text{Curvature}}, \tag{5.85}$$

since $\Omega_{m,0} = \Omega_0$ in such a universe. Suppose you are in a universe that is currently expanding ($H_0 > 0$) and contains nothing but nonrelativistic matter. If you ask the question, "Will the universe ever cease to expand?" then Equation 5.85 enables you to answer that question. For the universe to cease expanding, there must be some moment at which $H(t) = 0$. Since the first term on the right-hand side of

[9] Interestingly, this quote is from Eliot's poem *The Hollow Men*, written, for the most part, in 1924, the year when Friedmann published his second paper on the expansion of the universe. However, this coincidence seems to be just that – a coincidence. Eliot did not keep up to date on the technical literature of cosmology.

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Equation 5.85 is always positive, $H(t) = 0$ requires the second term on the right-hand side to be negative. This means that a matter-dominated universe will cease to expand if $\Omega_0 > 1$, and hence $\kappa = +1$. At the time of maximum expansion, $H(t) = 0$ and thus

$H(t)=0 \rightarrow \left(\frac{\Omega_0}{a^3} = \frac{\Omega_0 - 1}{a^2}\right) \times a^2$

$$0 = \frac{\Omega_0}{a_{\max}^3} + \frac{1-\Omega_0}{a_{\max}^2}. \tag{5.86}$$

The scale factor at the time of maximum expansion will therefore be

$\frac{\Omega_0}{a} = \Omega_0 - 1 \Longrightarrow$

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1}, \tag{5.87}$$

where Ω_0, remember, is the density parameter as measured at a scale factor $a = 1 \equiv a(t_0)$

Note that in Equation 5.85, the Hubble parameter enters only as H^2. Thus, the contraction phase, after the universe reaches maximum expansion, is just the time reversal of the expansion phase. (More precisely, the contraction is a perfect time reversal of the expansion only when the universe is perfectly homogeneous and the expansion is perfectly adiabatic, or entropy-conserving. In a real, lumpy universe, entropy is not conserved on small scales. Stars, for instance, generate entropy as they emit photons. During the contraction phase of an $\Omega_0 > 1$ universe, small-scale entropy-producing processes will NOT be reversed. Stars will not absorb the photons they previously emitted; people will not live backward from grave to cradle.) Eventually, the $\Omega_0 > 1$ universe will collapse down to $a = 0$, in an event sometimes called the "Big Crunch," after a finite time $t = t_{\text{crunch}}$. A matter-dominated universe with $\Omega_0 > 1$ not only has finite spatial extent, but also has a finite duration in time; just as it began in a hot, dense state, so it will end in a hot, dense state.

A matter-dominated universe with $\Omega_0 > 1$ will expand to a maximum scale factor $a_{\max}$, then collapse in a Big Crunch. What is the ultimate fate of a matter-dominated universe with $\Omega_0 < 1$ and $\kappa = -1$? In the Friedmann equation for such a universe (Equation 5.85), *both* terms on the right-hand side are positive. Thus if such a universe is expanding at a time $t = t_0$, it will continue to expand forever. At early times, when the scale factor is small ($a \ll \Omega_0/[1 - \Omega_0]$), the matter term of the Friedmann equation will dominate, and the scale factor will grow at the rate $a \propto t^{2/3}$. Ultimately, however, the density of matter will be diluted far below the critical density, and the universe will expand like the negatively curved empty universe, with $a \propto t$.

If a universe contains nothing but matter, its curvature, its density, and its ultimate fate are closely linked, as shown in Table 5.1. At this point, the obligatory quote is from Robert Frost: "Some say the world will end in fire / Some say in ice."[10] In a matter-dominated universe, if the density is greater than the critical

[10] This is from Frost's poem *Fire and Ice*, first published in Harper's Magazine in December 1920. Unlike T. S. Eliot, Frost was keenly interested in astronomy, and frequently wrote poems on astronomical themes.

Table 5.1 Curved, matter-dominated universes.

Density	*Curvature*	*Ultimate fate*
$\Omega_0 < 1$	$\kappa = -1$	Big Chill ($a \propto t$)
$\Omega_0 = 1$	$\kappa = 0$	Big Chill ($a \propto t^{2/3}$)
$\Omega_0 > 1$	$\kappa = +1$	Big Crunch

density, the universe will end in a fiery Big Crunch; if the density is less than or equal to the critical density, the universe will end in an icy Big Chill.

In a curved universe containing only matter, the scale factor $a(t)$ can be computed explicitly. The Friedmann equation can be written in the form

(5.85) $\times a^2$

$$\left(\frac{1}{H_0}\frac{da}{dt}\right)^2 = \frac{\dot{a}^2}{H_0^2} = \frac{\Omega_0}{a} + (1 - \Omega_0), \tag{5.88}$$

so the age t of the universe at a given scale factor a is given by the integral

(5.83) FOR {MATTER & CURVATURE ONLY}

$$H_0 t = \int_0^a \frac{da}{[\Omega_0/a + (1 - \Omega_0)]^{1/2}}. \tag{5.89}$$

When $\Omega_0 \neq 1$, the solution to this integral is most compactly written in a parametric form. The solution when $\Omega_0 > 1$ is ($\kappa = +1$)

$\boxed{\text{H/W 6.7a}}$

* SHOW & DISCUSS (5.89), (5.90), (5.91), (5.92)

* SKETCH $a(t)$

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta) \tag{5.90}$$

and

θ = A TIME-LIKE COORDINATE WITH $0 \leq \theta \leq 2\pi$ FOR $\kappa = +1$

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta), \tag{5.91}$$

where the parameter θ runs from 0 to 2π. Given this parametric form, the time that elapses between the Big Bang at $\theta = 0$ and the Big Crunch at $\theta = 2\pi$ can be computed as

$$t(a_{max}) = \frac{\frac{1}{2}\pi\,\Omega_0}{H_0(\Omega_0 - 1)^{3/2}} \Longleftarrow \quad t_{\text{crunch}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}. \tag{5.92}$$

A plot of a versus t in the case $\Omega_0 = 1.1$ is shown as the dotted line in Figure 5.4. The $a \propto t^{2/3}$ behavior of an $\Omega_0 = 1$ universe is shown as the solid line.

The solution of Equation 5.89 for the case $\Omega_0 < 1$ can be written in parametric form as

$\kappa = -1$

$\boxed{\text{H/W 6.7b}}$

* SHOW & DISCUSS (5.93), (5.94)

* SKETCH $a(t)$

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0}(\cosh\eta - 1) \tag{5.93}$$

and

η IS A TIME-LIKE COORDINATE ($0 \leq \eta \leq \infty$) FOR $\kappa = -1$

$$t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1 - \Omega_0)^{3/2}}(\sinh\eta - \eta), \tag{5.94}$$

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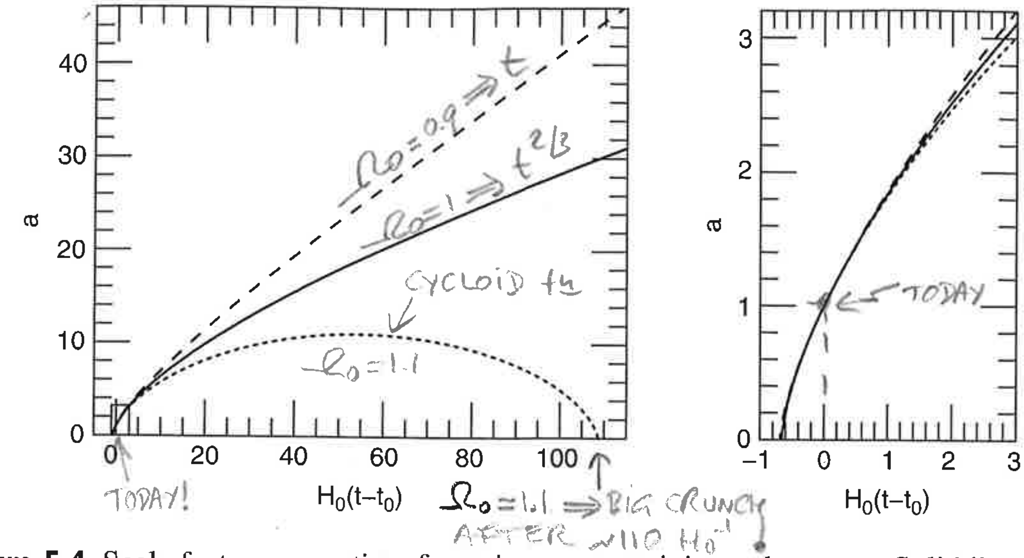

$\forall$ models behave like $a(t) \propto t^{2/3}$ for $t \downarrow 0$ (EdS)

Figure 5.4 Scale factor versus time for universes containing only matter. Solid line: $a(t)$ for a universe with $\Omega_0 = 1$ (flat). Dashed line: $a(t)$ for a universe with $\Omega_0 = 0.9$ (negatively curved). Dotted line: $a(t)$ for a universe with $\Omega_0 = 1.1$ (positively curved). The right panel is a blow-up of the small rectangle near the lower left corner of the left panel.

where the parameter η runs from 0 to infinity. A plot of a versus t in the case $\Omega_0 = 0.9$ is shown as the dashed line in Figure 5.4. Although the ultimate fate of an $\Omega_0 = 0.9$ universe is very different from that of an $\Omega_0 = 1.1$ universe, as shown graphically in the left panel of Figure 5.4, it is very difficult, at $t = t_0$, to tell a universe with Ω_0 slightly less than one from that with Ω_0 slightly greater than one. As shown in the right panel of Figure 5.4, the scale factors of the $\Omega_0 = 1.1$ universe and the $\Omega_0 = 0.9$ universe start to diverge significantly only after a Hubble time or more.

Scientists sometimes joke that they are searching for a theory of the universe that is compact enough to fit on the front of a T-shirt. If the energy content of the universe were contributed almost entirely by nonrelativistic matter, then an appropriate T-shirt slogan would be:

DENSITY
IS
DESTINY!

(ONLY FOR $\Lambda = 0$)

If the density of matter is less than the critical value, then the destiny of the universe is an ever-expanding Big Chill; if the density is greater than the critical value, then the destiny is a recollapsing Big Crunch. Like all terse summaries of complex concepts, the slogan "Density is Destiny!" requires a qualifying footnote. In this case, the required footnote is "*if $\Lambda = 0$." If the universe has a cosmological constant (or more generally, any component with $w < -1/3$), then the equation Density = Destiny = Curvature no longer applies.

5.4.2 Matter + Lambda

Consider a universe that is spatially flat, but contains both matter and a cosmological constant. (Such a universe is of particular interest to us, since it is a close approximation to our own universe at the present day.) If, at a given time $t = t_0$, the density parameter in matter is $\Omega_{m,0}$ and the density parameter in a cosmological constant Λ is $\Omega_{\Lambda,0}$, the requirement that space be flat tells us that

$$\Omega_{\Lambda,0} + \Omega_{m,0} = 1 \Longleftrightarrow \quad \Omega_{\Lambda,0} = 1 - \Omega_{m,0}, \qquad (K=0) \tag{5.95}$$

and the Friedmann equation for the flat "matter plus lambda" universe reduces to

(5.81) WITH MATTER AND LAMBDA ONLY $\Rightarrow$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0}). \qquad (K=0) \tag{5.96}$$

The first term on the right-hand side of Equation 5.96 represents the contribution of matter, and is always positive. The second term represents the contribution of a cosmological constant; it is positive if $\Omega_{m,0} < 1$, implying $\Omega_{\Lambda,0} > 0$, and negative if $\Omega_{m,0} > 1$, implying $\Omega_{\Lambda,0} < 0$. Thus, a flat universe with $\Omega_{\Lambda,0} > 0$ will continue to expand forever if it is expanding at $t = t_0$; this is another example of a Big Chill universe. In a universe with $\Omega_{\Lambda,0} < 0$, however, the negative cosmological constant provides an *attractive* force, not the repulsive force of a positive cosmological constant. A flat universe with $\Omega_{\Lambda,0} < 0$ will cease to expand at a maximum scale factor

(5.87) WITH $\frac{H(t)}{H_0} = 0$ $\Rightarrow$

$$a_{\max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\right)^{1/3}, \tag{5.97}$$

and will collapse back down to $a = 0$ at a cosmic time

(5.92) $t_{crunch} = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$

$$t_{\rm crunch} = \frac{2\pi}{3H_0} \frac{1}{\sqrt{\Omega_{m,0} - 1}}. \tag{5.98}$$

For a given value of H_0, the larger the value of $\Omega_{m,0}$, the shorter the lifetime of the universe. For a flat, $\Omega_{\Lambda,0} < 0$ universe, the Friedmann equation can be integrated to yield the analytic solution

H/W 6.8a FOR $\Omega_{\Lambda,0} < 0$
*SHOW 5.97, 5.98, 5.99
[PREFER YOU DO H/W 6.8b]
*PLOT a(t)

$$H_0 t = \frac{2}{3\sqrt{\Omega_{m,0} - 1}} \sin^{-1}\left[\left(\frac{a}{a_{\max}}\right)^{3/2}\right]. \tag{5.99}$$

A plot of a versus t in the case $\Omega_{m,0} = 1.1$, $\Omega_{\Lambda,0} = -0.1$ is shown as the dotted line in Figure 5.5. The $a \propto t^{2/3}$ behavior of an $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$ universe is shown, for comparison, as the solid line. A flat universe with $\Omega_{\Lambda,0} < 0$ ends in a Big Crunch, reminiscent of that for a positively curved, matter-only universe. However, with a negative cosmological constant providing an attractive force, the lifetime of a flat universe with $\Omega_{\Lambda,0} < 0$ is exceptionally short. For instance, we have seen that a positively curved universe with $\Omega_{m,0} = 1.1$ undergoes a Big

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Crunch after a lifetime $t_{\rm crunch} \approx 110 H_0^{-1}$ (Figure 5.4). However, a flat universe with $\Omega_{m,0} = 1.1$ and $\Omega_{\Lambda,0} = -0.1$ has a lifetime of only $t_{\rm crunch} \approx 7H_0^{-1}$.

Although a negative cosmological constant is permitted by the laws of physics, it appears that we live in a universe with a positive cosmological constant. In a flat universe with $\Omega_{m,0} < 1$ and $\Omega_{\Lambda,0} > 0$, the density contributions of matter and the cosmological constant are equal at the scale factor (Equation 5.21):

$$(5.21) \Rightarrow \qquad a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \simeq \left(\frac{0.32}{0.68}\right)^{1/3} \simeq 0.78 \qquad (5.100) \Rightarrow z_{m,\Lambda} \simeq 0.286 \; (T_{m,\Lambda} \simeq 3.43\,\text{Gyr})$$

For a flat, $\Omega_{\Lambda,0} > 0$ universe, the Friedmann equation can be integrated to yield the analytic solution

H/W 6.8b / FOR $\Omega_{\Lambda,0} > 0$

*SHOW (5.101)–(5.104)

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^3}\,\right]. \qquad (5.101)$$

*PLOT $a(t)$

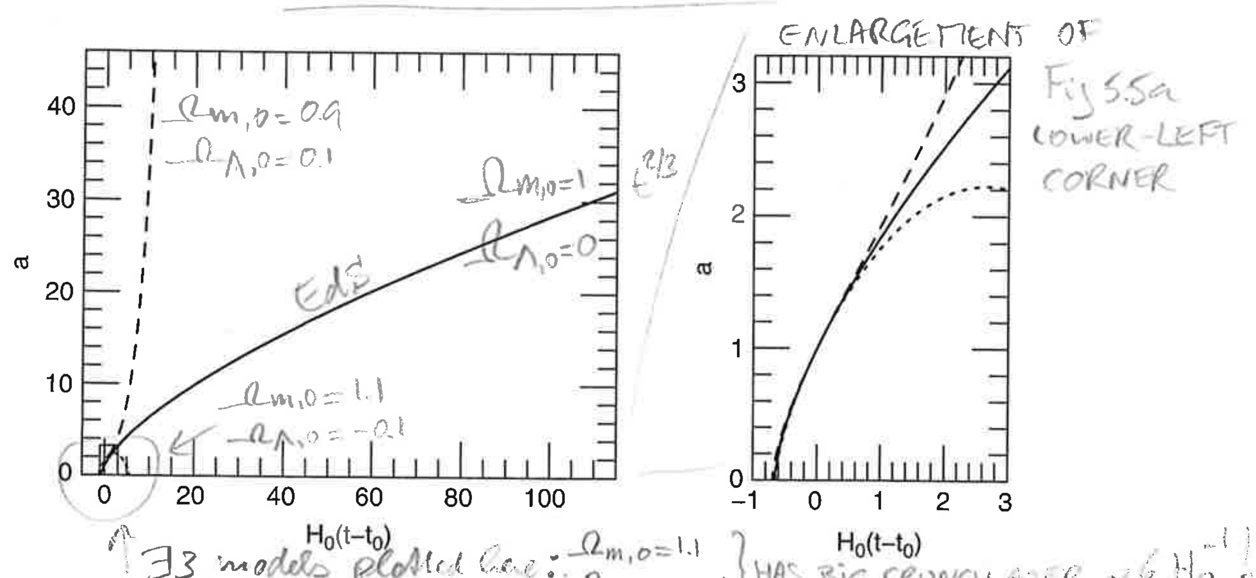
A plot of a versus t in the case $\Omega_{m,0} = 0.9$, $\Omega_{\Lambda,0} = 0.1$ is shown as the dashed line in Figure 5.5. At early times, when $a \ll a_{m\Lambda}$, Equation 5.101 reduces to the relation

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\,H_0 t\right)^{2/3} \simeq \left(\frac{t}{("H_0" \simeq 57)^{-1}}\right)^{2/3} \qquad (5.102)$$

FOR TRUE $H_0 = 67$, $\Omega_{m,0} = 0.32$ ($\Omega_\Lambda = 0.68$)

giving the $a \propto t^{2/3}$ dependence required for a flat, matter-dominated universe. At late times, when $a \gg a_{m\Lambda}$, Equation 5.101 reduces to

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{1-\Omega_{m,0}}\,H_0 t), \qquad (5.103)$$



3 models plotted here: $\Omega_{m,0} = 1.1$, $\Omega_{\Lambda,0} = -0.1$ } HAS BIG CRUNCH AFTER $\sim 6 H_0^{-1}$!

Figure 5.5 Scale factor versus time for flat universes containing both matter and a cosmological constant. Solid line: $a(t)$ for a universe with $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$. Dashed line: $a(t)$ for a universe with $\Omega_{m,0} = 0.9$, $\Omega_{\Lambda,0} = 0.1$. Dotted line: $a(t)$ for a universe with $\Omega_{m,0} = 1.1$, $\Omega_{\Lambda,0} = -0.1$. The right panel is a blow-up of the small rectangle near the lower left corner of the left panel.

giving the $a \propto e^{Kt}$ dependence required for a flat, lambda-dominated universe. Suppose you are in a flat universe containing nothing but matter and a cosmological constant; if you measure H_0 and $\Omega_{m,0}$, then Equation 5.101 tells you that the age of the universe is

(5.100) (5.101) ⇒

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right]. \tag{5.104}$$

If we approximate our own universe as having $\Omega_{m,0} = 0.31$ (0.32) and $\Omega_{\Lambda,0} = 0.69$ (0.68), we find that its current age is

0.947

$$t_0 = 0.955H_0^{-1} = 13.74 \pm 0.40\,\text{Gyr}, \tag{5.105}$$

67 (13.82)

assuming $H_0 = 68 \pm 2\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. (We'll see in Section 5.5 that ignoring the radiation content of the universe has an insignificant effect on our estimate of t_0.) The age at which matter and the cosmological constant had equal energy density was

at $z_{m\Lambda} \simeq 0.28$

$$t_{m\Lambda} = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln[1+\sqrt{2}] = 0.707H_0^{-1} = 10.17 \pm 0.30\,\text{Gyr}. \tag{5.106}$$

Thus, if our universe is well described by the Benchmark Model, with $\Omega_{m,0} = 0.31$ and $\Omega_{\Lambda,0} \approx 0.69$, then the cosmological constant has been the dominant component of the universe for the last 3.6 billion years or so. (≃ Age of Earth since epoch of max impacts)

3.4

5.4.3 Matter + Curvature + Lambda

$\forall\, K = 0, \pm 1$

By choosing different values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, without constraining the universe to be flat, we can create model universes with scale factors $a(t)$ that exhibit very interesting behavior. Start by writing down the Friedmann equation for a curved universe with both matter and a cosmological constant:

$\Omega_{r,0} \equiv 0$

(5.81) + $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0}$

$$\frac{H^2}{H_0^2} = \underbrace{\frac{\Omega_{m,0}}{a^3}}_{\text{MATTER}} + \underbrace{\frac{\overbrace{1-\Omega_{m,0}-\Omega_{\Lambda,0}}^{(1-\Omega_0)}}{a^2}}_{\text{CURVATURE}} + \underbrace{\overbrace{\Omega_{\Lambda,0}}^{1-\Omega_{m,0}}}_{\text{LAMBDA}}. \tag{5.107}$$

If $\Omega_{m,0} > 0$ and $\Omega_{\Lambda,0} > 0$, then both the first and last term on the right-hand side of Equation 5.107 are positive. However, if $\Omega_{m,0} + \Omega_{\Lambda,0} > 1$, so that the universe is positively curved, then the central term on the right-hand side is negative. As a result, for some choices of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, the value of H^2 will be positive for small values of a (where matter dominates) and for large values of a (where Λ dominates), but will be negative for intermediate values of a (where the curvature term dominates). Since negative values of H^2 are unphysical, this means that these universes have a forbidden range of scale factors. Suppose such a universe starts out with $a \gg 1$ and $H < 0$; that is, it is contracting from a low-density, Λ-dominated state. As the universe contracts, however, the negative curvature term in Equation 5.107 becomes dominant, causing the contraction to stop at

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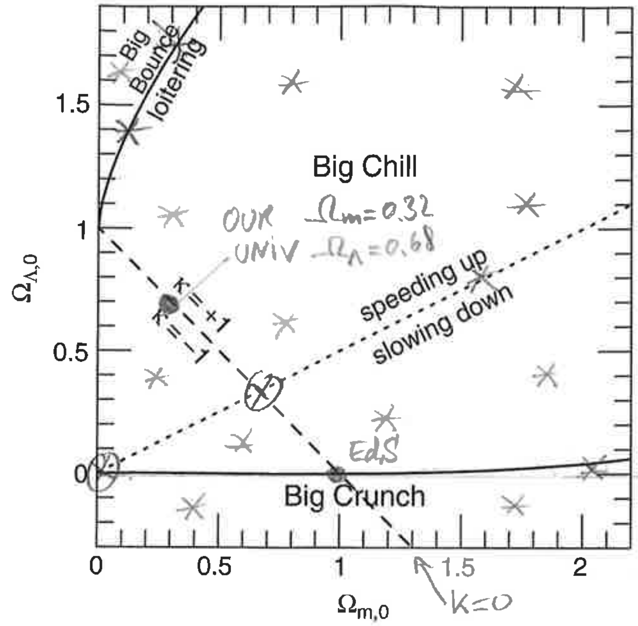

If you choose proving Fig 5.6 in detail in a term project, you need to solve the Friedmann Equations in all relevant areas + show plots of $a(t)$, i.e have solutions for all $(\Omega_m, \Omega_\Lambda)$ locations $\ast$

Figure 5.6 Properties of universes containing matter and a cosmological constant. The dashed line indicates flat universes ($\kappa = 0$). The dotted line indicates universes that are not accelerating today ($q_0 = 0$ at $a = 1$). Also shown are the regions where the universe has a "Big Chill" expansion ($a \to \infty$ as $t \to \infty$), a "Big Crunch" recollapse ($a \to 0$ as $t \to t_{\text{crunch}}$), a loitering phase ($a \approx$ constant for an extended period), or a "Big Bounce" ($a = a_{\min} > 0$ at $t = t_{\text{bounce}}$).

a minimum scale factor $a = a_{\min}$, and then expand outward again in a "Big Bounce." Thus, it is possible to have a universe that expands outward at late times, but never had an initial Big Bang, with $a = 0$ at $t = 0$.

Another possibility, if the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are chosen just right, is a "loitering" universe. Such a universe starts in a matter-dominated state, expanding outward with $a \propto t^{2/3}$. Then, however, it enters a stage (called the loitering stage) in which a is very nearly constant for a long period of time. During this time it is almost – but not quite – Einstein's static universe. After the loitering stage, the cosmological constant takes over, and the universe starts to expand exponentially.[11]

Figure 5.6 shows the general behavior of the scale factor $a(t)$ as a function of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. In the region labeled "Big Crunch," the universe starts with $a = 0$ at $t = 0$, reaches a maximum scale factor $a_{\max}$, then recollapses to $a = 0$ at a finite time $t = t_{\text{crunch}}$. Note that Big Crunch universes can be positively curved, negatively curved, or flat. In the region labeled "Big Chill," the universe starts with $a = 0$ at $t = 0$, then expands outward forever, with $a \to \infty$ as $t \to \infty$. Like Big Crunch universes, Big Chill universes can have any sign for their curvature.

[11] A loitering universe is sometimes referred to as a *Lemaître universe*, since Georges Lemaître discussed, in his 1927 paper on the expanding universe, the possibility of a loitering stage extending into the indefinitely distant past.

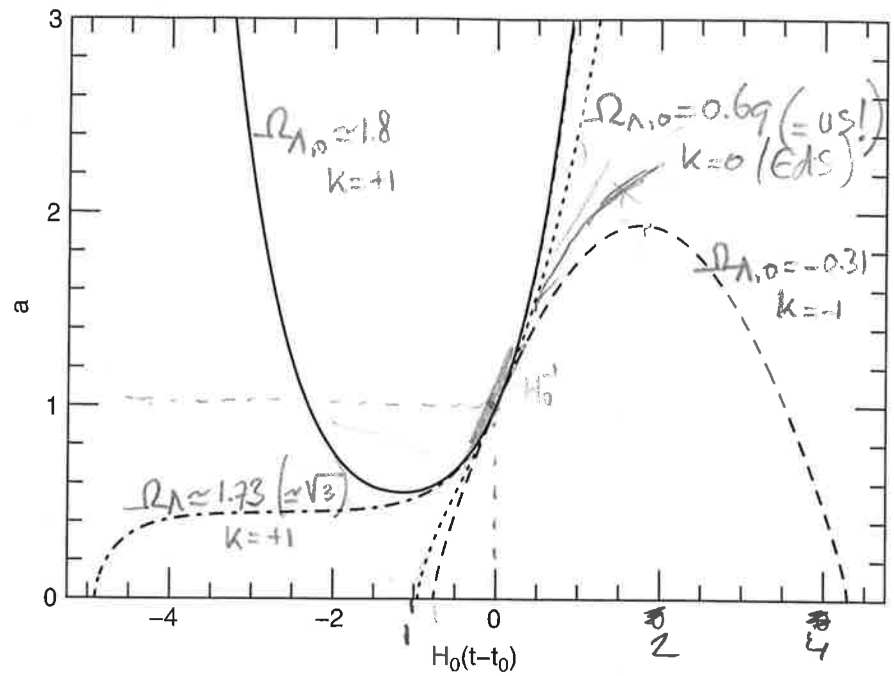

Figure 5.7 Scale factor versus time in four different universes, each with $\Omega_{m,0} = 0.31$. Dotted line: a flat "Big Chill" universe ($\Omega_{\Lambda,0} = 0.69$, $\kappa = 0$). Dashed line: a "Big Crunch" universe ($\Omega_{\Lambda,0} = -0.31$, $\kappa = -1$). Dot-dash line: a loitering universe ($\Omega_{\Lambda,0} = 1.7289$, $\kappa = +1$). Solid line: a "Big Bounce" universe ($\Omega_{\Lambda,0} = 1.8$, $\kappa = +1$).

In the region labeled "Big Bounce," the universe starts in a contracting state, reaches a minimum scale factor $a = a_{\min} > 0$ at some time t_{bounce}, then expands outward forever, with $a \to \infty$ as $t \to \infty$. Universes that fall just below the dividing line between Big Bounce universes and Big Chill universes are loitering universes. The closer such a universe lies to the Big Bounce–Big Chill dividing line in Figure 5.6, the longer its loitering stage lasts.

To illustrate the possible types of expansion and contraction, Figure 5.7 shows $a(t)$ for a set of four model universes. Each of these universes has the same current density parameter for matter: $\Omega_{m,0} = 0.31$, measured at $t = t_0$ and $a = 1$. These universes cannot be distinguished from each other by measuring their current matter density and Hubble constant. Nevertheless, thanks to their different values for the cosmological constant, they have very different pasts and very different futures. The dotted line in Figure 5.7 shows the scale factor $a(t)$ for a universe with $\Omega_{\Lambda,0} = 0.69$; this universe is spatially flat, and is destined to end in an exponentially expanding Big Chill. The dashed line shows $a(t)$ for a universe with $\Omega_{\Lambda,0} = -0.31$; this universe has an energy density of zero, and is negatively curved. After expanding to a maximum scale factor $a_{\max} \approx 1.93$, it will recollapse in a Big Crunch. The dot-dash line shows the scale factor for a universe with $\Omega_{\Lambda,0} = 1.7289$; this is a positively curved loitering universe, which spends a long time with a scale factor $a \approx a_{\text{loiter}} \approx 0.45$. Finally, the solid line shows a universe with $\Omega_{\Lambda,0} = 1.8$. This universe lies above the Big Chill–Big Bounce dividing line

HP/TP
As part of
Show Also

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in Figure 5.6; it is a positively curved universe that "bounced" at a scale factor $a = a_{\text{bounce}} \approx 0.552$. If we lived in this Big Bounce universe, the largest redshift we could see would be $z_{\max} = 1/a_{\text{bounce}} - 1 \approx 0.81$. Extremely distant light sources would actually be blueshifted. !!

(5.18)

$\frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{9\times10^{-5}}{0.32} \approx 2.8\times10^{-4}$

$a_{rm} \approx 3550$

5.4.4 Radiation + Matter

In our universe, radiation–matter equality took place at a scale factor $a_{rm} \equiv \Omega_{r,0}/\Omega_{m,0} \approx 2.9 \times 10^{-4}$. At scale factors $a \ll a_{rm}$, the universe is well described by a flat, radiation-only model, as described in Section 5.3.2. At scale factors $a \sim a_{rm}$, the universe is better described by a flat model containing both radiation and matter. The Friedmann equation around the time of radiation–matter equality can be written in the approximate form

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}. \qquad (\times a^4)^{1/2} \tag{5.108}$$

This can be rearranged in the form

$a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.8\times10^{-4} = \frac{1}{3550}$

$\boxed{\text{H/W 6.9}}$

* Show (5.109)–(5.113)

* Plot $a(t)$ for $a \geq a_{rm}$, $a \leq a_{rm}$

$$H_0 dt = \frac{a da}{\Omega_{r,0}^{1/2}} \left[1 + \frac{a}{a_{rm}}\right]^{-1/2}. \tag{5.109}$$

Integration yields a fairly simple relation for t as a function of a during the epoch when only radiation and matter are significant:

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right]. \tag{5.110}$$

In the limit $a \ll a_{rm}$, this gives the appropriate result for the radiation-dominated phase of evolution,

$$a \approx \left(2\sqrt{\Omega_{r,0}}\, H_0 t\right)^{1/2} \qquad [a \ll a_{rm}]. \tag{5.111}$$

In the limit $a \gg a_{rm}$ (but before curvature or Λ contributes significantly to the Friedmann equation), the approximate result for $a(t)$ becomes

$$a \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t\right)^{2/3} \qquad [a \gg a_{rm}]. \tag{5.112}$$

The time of radiation–matter equality, t_{rm}, can be found by setting $a = a_{rm}$ in Equation 5.110:

$$t_{rm} = \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}} H_0^{-1} \approx 0.391 \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} H_0^{-1}. \tag{5.113}$$

For the Benchmark Model, with $\Omega_{r,0} = 9.0 \times 10^{-5}$, $\Omega_{m,0} = 0.31$ (0.32), and $H_0^{-1} =$ 14.4 Gyr, the time of radiation–matter equality was

$$t_{rm} = 3.47 \times 10^{-6} H_0^{-1} = 50\,000\,\text{yr}. \tag{5.114}$$

3.26 47,000

The epoch when the universe was radiation-dominated was only about 50 millennia long. This is sufficiently brief that it justifies our ignoring the effects of radiation when computing the age of the universe. The age $t_0 = 0.955 H_0^{-1} =$ 13.7 Gyr that we computed in Section 5.4.2 (ignoring radiation) would only be altered by a few parts per million if we included the effects of radiation. This minor correction is dwarfed by the uncertainty in the value of H_0^{-1}. $0.947 H_0^{-1}$ 13.8

5.5 Benchmark Model

The Benchmark Model, which we have adopted as a good fit to the currently available observational data, is spatially flat, and contains radiation, matter, and a cosmological constant. Some of its properties are listed, for ready reference, in Table 5.2. The Hubble constant of the Benchmark Model is assumed to be $H_0 = 68\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. The radiation in the Benchmark Model consists of photons and neutrinos. The photons are assumed to be provided solely by a cosmic microwave background with current temperature $T_0 = 2.7255$ K and density parameter $\Omega_{\gamma,0} = 5.35 \times 10^{-5}$. The energy density of the cosmic neutrino background is theoretically calculated to be 68.1% of that of the cosmic microwave background, as long as neutrinos are relativistic. If a neutrino has a nonzero mass m_ν, Equation 5.17 tells us that it defects from the "radiation" column to

$H_0 = 66.9$

Table 5.2 Properties of the Benchmark Model.

List of ingredients		
Photons:	$\Omega_{\gamma,0} = 5.35 \times 10^{-5}$	
Neutrinos:	$\Omega_{\nu,0} = 3.65 \times 10^{-5}$	
Total radiation:	$\Omega_{r,0} = 9.0 \times 10^{-5}$	
Baryonic matter:	$\Omega_{\text{bary},0} = 0.048$	
Nonbaryonic dark matter:	$\Omega_{\text{dm},0} = 0.262$	
Total matter:	$\Omega_{m,0} = 0.31$ 0.32	
Cosmological constant:	$\Omega_{\Lambda,0} \approx 0.69$ 0.68	
Important epochs		
Radiation–matter equality:	$a_{rm} = 2.9 \times 10^{-4} \simeq 1/3500$	$t_{rm} = 0.050$ Myr
Matter–lambda equality:	$a_{m\Lambda} = 0.77 \simeq 1/0.283$	$t_{m\Lambda} = 10.2$ Gyr (3.6 Gyr ago)
Now:	$a_0 = 1$	$t_0 = 13.7$ Gyr 13.8

$\Omega_{dark} = 0.262 \simeq 5.5$
Ω_b 0.048

2018 See Planck 2016 cosmology paper on our URL for latest values

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the "matter" column when the scale factor is $a \sim 5 \times 10^{-4}\,\text{eV}/(m_\nu c^2)$. The matter content of the Benchmark Model consists partly of baryonic matter (that is, matter composed of protons and neutrons, with associated electrons) and partly of nonbaryonic dark matter. The baryonic material that we are familiar with from our everyday existence has a density parameter $\Omega_{\text{bary},0} \approx 0.048$ today. The density parameter of the nonbaryonic dark matter is over five times greater: $\Omega_{\text{dm},0} \approx 0.262$. The bulk of the energy density in the Benchmark Model, however, is not provided by radiation or matter, but by a cosmological constant, with $\Omega_{\Lambda,0} = 1 - \Omega_{m,0} - \Omega_{r,0} \approx 0.69$. (0.68)

With $\Omega_{r,0}$, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$ known, the scale factor $a(t)$ can be computed numerically using the Friedmann equation, in the form of Equation 5.81. Figure 5.8 shows the scale factor, thus computed, for the Benchmark Model. Note that the transition from the $a \propto t^{1/2}$ radiation-dominated phase to the $a \propto t^{2/3}$ matter-dominated phase is not an abrupt one; neither is the later transition from the matter-dominated phase to the exponentially growing lambda-dominated phase. One curious feature of the Benchmark Model illustrated vividly in Figure 5.8 is that we are living very close to the time of matter–lambda equality (at least, as plotted on a logarithmic scale). A COINCIDENCE ?!!

Once $a(t)$ is known, other properties of the Benchmark Model can be computed readily. For instance, the left panel of Figure 5.9 shows the current proper distance to a galaxy with redshift z. The heavy solid line is the result for the

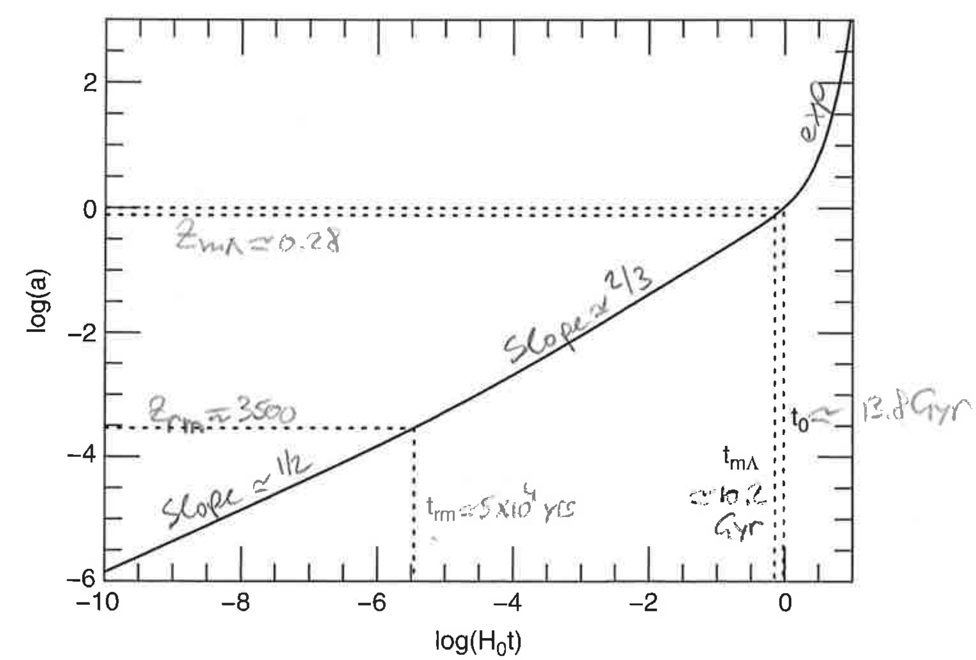

Figure 5.8 The scale factor a as a function of time t (measured in units of the Hubble time), computed for the Benchmark Model. The dotted lines indicate the time of radiation–matter equality, $a_{rm} = 2.9 \times 10^{-4}$, the time of matter–lambda equality, $a_{m\Lambda} = 0.77$, and the present moment, $a_0 = 1$.

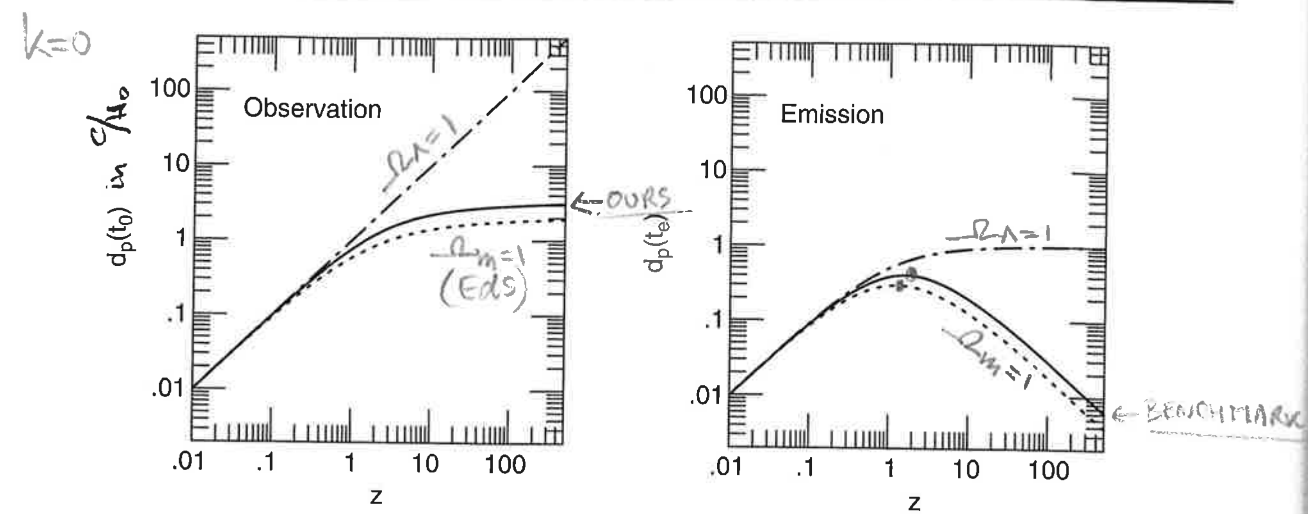

Figure 5.9 The proper distance to a light source with redshift z, in units of the Hubble distance, c/H_0. The left panel shows the distance at the time of observation; the right panel shows the distance at the time of emission. The bold solid line indicates the Benchmark Model. For comparison, the dot-dash line indicates a flat, lambda-only universe, and the dotted line a flat, matter-only universe.

Benchmark Model; for purposes of comparison, the result for a flat lambda-only universe is shown as a dot-dash line and the result for a flat matter-only universe is shown as the dotted line. In the limit $z \to \infty$, the proper distance $d_p(t_0)$ approaches a limiting value $d_p \to 3.20c/H_0$, in the case of the Benchmark Model. Thus, the Benchmark Model has a finite horizon distance,

$$d_{\rm hor}(t_0) = 3.20c/H_0 = 3.35ct_0 = 14\,000\,{\rm Mpc}. \tag{5.115}$$

If the Benchmark Model is a good description of our own universe, then we can't see objects more than 14 gigaparsecs away because light from them has not yet had time to reach us. The right panel of Figure 5.9 shows $d_p(t_e)$, the distance to a galaxy with observed redshift z at the time the observed photons were emitted. For the Benchmark Model, $d_p(t_e)$ has a maximum for galaxies with redshift $z = 1.6$, 1.65
where $d_p(t_e) = 0.405c/H_0 \simeq 0.405\, R_0$

When astronomers observe a distant galaxy, they ask the related, but not identical, questions, "How far away is that galaxy?" and "How long has the light from that galaxy been traveling?" In the Benchmark Model, or any other model, we can answer the question "How far away is that galaxy?" by computing the proper distance $d_p(t_0)$. We can answer the question "How long has the light from that galaxy been traveling?" by computing the *lookback time*. If light emitted at time t_e is observed at time t_0, the lookback time is simply $t_0 - t_e$. In the
FOR $z \ll 1$ limit of very small redshifts, $t_0 - t_e \approx z/H_0$. However, as shown in Figure 5.10, at larger redshifts the relation between lookback time and redshift becomes
$\tau = t_0 - t_e \simeq \frac{z}{H_0}$ nonlinear. The exact dependence of lookback time on redshift depends on the cosmological model used. For example, consider a galaxy with redshift $z = 2$.

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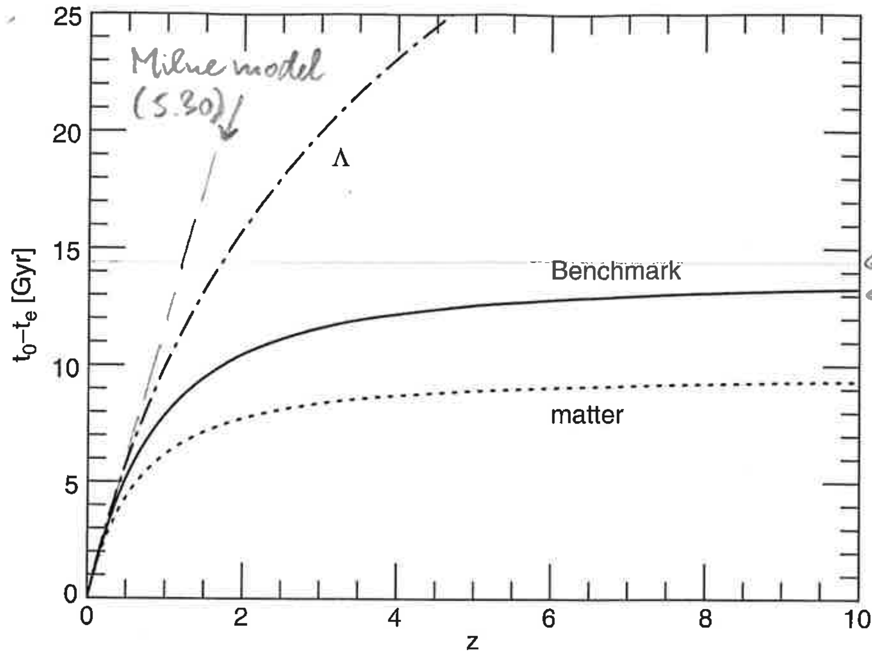

(5.105)
$\Downarrow$
$\leftarrow H_0^{-1} \simeq 14.4$ Gyr
$\leftarrow 0.947\, H_0^{-1} \simeq 13.8$ Gyr

Figure 5.10 The lookback time, $t_0 - t_e$, for galaxies with observed redshift z. The Hubble time is assumed to be $H_0^{-1} = 14.4$ Gyr. The bold solid line shows the result for the Benchmark Model. For comparison, the dot-dash line indicates a flat, lambda-only universe, and the dotted line a flat, matter-only universe.

$z=2$

In the Benchmark Model, the lookback time to that galaxy is 10.5 Gyr; we are seeing a redshifted image of that galaxy as it was 10.5 billion years ago. In a flat, lambda-only universe, however, the lookback time to a $z = 2$ galaxy is 15.8 Gyr, assuming $H_0^{-1} = 14.4$ Gyr. In a flat, matter-dominated universe, the lookback time to a $z = 2$ galaxy is a mere 7.7 Gyr, with the same assumed Hubble constant.

The most distant galaxies that have been observed (at the time of writing) are at a redshift $z \approx 10$. Consider such a high-redshift galaxy. Using the Benchmark Model, we find that the current proper distance to a galaxy with $z = 10$ is $d_p(t_0) = 2.18c/H_0 = 9500$ Mpc, about two-thirds of the current horizon distance. The proper distance at the time the light was emitted was $d_p(t_e) = d_p(t_0)/(1 + z) = 0.20c/H_0 = 870$ Mpc. The light we observe now was emitted when the age of the universe was $t_e = 0.033H_0^{-1} = 0.47$ Gyr; this is less than 4% of the universe's current age, $t_0 = 0.955H_0^{-1} = 13.74$ Gyr. The lookback time to a $z = 10$ galaxy in the Benchmark Model is thus $t_0 - t_e = 0.922H_0^{-1} = 13.27$ Gyr. Astronomers are fond of saying, "A telescope is a time machine."[12] As you look further and further out into the universe, to objects with larger and larger values of $d_p(t_0)$, you are looking back to objects with smaller and smaller values of t_e. When you observe a galaxy with a redshift $z = 10$, according to the Benchmark Model, you are glimpsing the universe as it was as a youngster, less than half a billion years old.

Use Ned Wright Cosmo Calc. in "Linux"

$\tau(z=10) = t_0 - t_e \simeq 13.3$ Gyr (0.5 Gyr AFTER BB)

[12] Or, as William Herschel phrased it over two centuries ago, "A telescope with a power of penetrating into space...has also, as it may be called, a power of penetrating into time past."