4

Cosmic Dynamics

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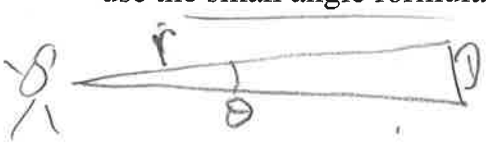
The idea that the universe could be curved, or non-Euclidean, long predates Einstein's theory of general relativity. As early as 1829, half a century before Einstein's birth, Nikolai Ivanovich Lobachevski, one of the founders of non-Euclidean geometry, proposed observational tests to demonstrate whether the universe was curved. In principle, measuring the curvature of the universe is simple; in practice, it is much more difficult. In principle, we could determine the curvature by drawing a really, really big triangle, and measuring the angles α, β, and γ at the vertices. Equations 3.22, 3.25, and 3.27 generalize to the equation

$$\alpha + \beta + \gamma = \pi + \frac{\kappa A}{R_0^2}, \tag{4.1}$$

$\kappa =$ +1 POSITIVE CURVATURE, 0 FLAT, −1 NEGATIVE CURVATURE

where A is the area of the triangle. Therefore, if $\alpha + \beta + \gamma > \pi$ radians, the universe is positively curved, and if $\alpha + \beta + \gamma < \pi$ radians, the universe is negatively curved. If, in addition, we measure the area of the triangle, we can determine the radius of curvature R_0. Unfortunately for this elegant geometric plan, the area of the biggest triangle we can draw is much smaller than R_0^2, and the deviation of $\alpha + \beta + \gamma$ from π radians would be too small to measure.

We can conclude from geometric arguments that if the universe is curved, it can't have a radius of curvature R_0 that is significantly smaller than the current Hubble distance, $c/H_0 \approx 4380\,\text{Mpc}$. To see why, consider a galaxy of diameter D that is at a distance r from the Earth. In a flat universe, in the limit $D \ll r$, we can use the small angle formula to compute the observed angular size α of the galaxy:



$$\theta = \alpha = \frac{D}{r}. \quad \text{EUCLIDEAN} \tag{4.2}$$

In a positively curved universe, the angular size is

See Eq (3.35) ⇒

$$\alpha_+ = \frac{D}{R_0 \sin(r/R_0)}. \tag{4.3}$$

When $r < \pi R_0$, then $\alpha_+ > D/r$, and the galaxy appears larger in size than it would in a flat universe. That is, in a positively curved universe, the curvature acts as a magnifying lens. Notice that the angular size α_+ blows up at $r = \pi R_0$; physically, this means that when a galaxy is at a distance corresponding to half the circumference of the universe, it fills the entire sky. No such bloated, highly-magnified galaxies are seen, even though we can see galaxies at distances as great as $r \sim c/H_0$. Thus, we conclude that if the universe is positively curved, it must have $\pi R_0 > c/H_0 \equiv R_H$

In a negatively curved universe, the observed angular size of the galaxy is

$$\alpha_- = \frac{D}{R_0 \sinh(r/R_0)} < \frac{D}{r} \,. \tag{4.4}$$

At a distance $r \gg R_0$, we can use the approximation $\sinh x \approx e^x/2$ to find

$$\alpha_- \approx \frac{2D}{R_0} \exp\left(-\frac{r}{R_0}\right) . \tag{4.5}$$

In a negatively curved universe, galaxies at a distance much greater than the radius of curvature R_0 appear exponentially tiny in angle. However, in our universe, galaxies are seen to be resolved in angle out to distances $r \sim c/H_0$. Thus, we conclude that if the universe is negatively curved, it must have $R_0 > c/H_0 \equiv R_H$

4.1 Einstein's Field Equation

In the 19th century, mathematicians such as Lobachevski were able to conceive of curved space. However, it wasn't until Einstein published his theory of general relativity in 1915 that anyone related the curvature of space (and time) to the physical content of the universe. The key equation of general relativity, which gives the mathematical relation between spacetime curvature and the energy density and pressure of the universe, is the field equation.

Einstein's field equation plays a role in general relativity that is analogous to the role played by Poisson's equation in Newtonian dynamics. Poisson's equation,

Eq.(3.5):

$$\boxed{\nabla^2 \Phi = 4\pi G \rho,} \tag{4.6}$$

tells you how to compute the gravitational potential Φ, given the mass density ρ of the material filling the universe. By taking the gradient of Φ, you determine the acceleration, and can then compute the trajectory of objects moving freely through space under the influence of gravity. Analogously, you can use Einstein's field equation to compute the curvature of spacetime, given the energy density ε, pressure P, and other properties of the material filling the universe. You can then compute the trajectory of a freely moving object by finding the appropriate geodesic in the curved spacetime.

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Einstein's field equation looks simple when it is written down:

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.} \tag{4.7}$$

The simplicity is deceptive, since the compact notation hides a great deal of necessary detail. The quantity $G_{\mu\nu}$ on the left-hand side of Equation 4.7 is the *Einstein tensor*, which is a 4×4 tensor that describes the curvature of spacetime at every location (t, x, y, z). It is a symmetric tensor, with $G_{\mu\nu} = G_{\nu\mu}$, so it has ten independent components. On the right-hand side of Equation 4.7, the quantity $T_{\mu\nu}$ is the *stress-energy tensor*, sometimes called the energy-momentum tensor; like the Einstein tensor, it is a 4×4 symmetric tensor.

The deceptively simple field equation, $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$, is actually a set of ten nonlinear second-order differential equations. Even without making the herculean effort to solve them exactly to find the curvature everywhere (and everywhen) in spacetime, we can make general statements about the properties of the solution. Since the ten differential equations are second order, this means that spacetime can have nonzero curvature even in spacetime neighborhoods where the stress-energy tensor $T_{\mu\nu}$ is zero. (Analogously, Poisson's equation is a second-order differential equation, and gravitational acceleration can be nonzero even in spatial neighborhoods where the mass density is zero.) Another property of second-order differential equations involving space and time is that they can yield propagating wave solutions, in which disturbances propagate through space as a function of time. Just as a time-varying electric dipole creates electromagnetic waves, a time-varying mass-energy quadrupole creates gravitational waves.[1]

In the most general case, the stress-energy tensor $T_{\mu\nu}$ can be very complicated, and difficult to calculate. However, things become much simpler if the universe is filled with a homogeneous and isotropic perfect gas. In that case, an observer who sees the universe expanding uniformly around her will measure an energy density $\varepsilon(t)$ and a pressure $P(t)$ for the ideal gas that are a function only of cosmic time; the observer will not measure any bulk velocity $\vec{u}$ for the gas, since that would break the isotropy. In this idealized case (which fortunately is a good approximation for our purposes), the stress-energy tensor $T_{\mu\nu}$ depends only on $\varepsilon(t)$ and $P(t)$. The metric describing the curvature of spacetime, in this case, is the homogeneous and isotropic Robertson–Walker metric (Equation 3.41):

(3.41)

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right], \tag{4.8}$$

[1] Einstein predicted the existence of gravitational waves in 1916. He then "un-predicted" them in 1936, when he erroneously thought they were a byproduct of the approximations he had made in 1916. In the 1950s, however, physicists "re-predicted" the existence of gravitational waves, which were finally detected by the Laser Interferometry Gravitational-wave Observatory on 14 Sept. 2015, just in time for the centenary of Einstein's prediction.

where

See Eq. (3.35):

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & (\kappa = +1) \\ r & (\kappa = 0) \\ R_0 \sinh(r/R_0) & (\kappa = -1). \end{cases} \tag{4.9}$$

Our remaining goal is to find how $a(t)$, κ, and R_0, the parameters that describe curvature, are linked to $\varepsilon(t)$ and $P(t)$, the parameters that describe the contents of the universe.

4.2 The Friedmann Equation

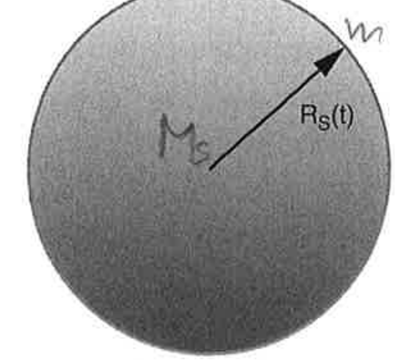
The equation that links together $a(t)$, κ, R_0, and $\varepsilon(t)$ is known as the *Friedmann equation*, after Alexander Friedmann, the Russian physicist who first derived the equation in 1922.[2] Friedmann actually started his scientific career as a meteorologist. Later, however, he taught himself general relativity, and used Einstein's field equation to describe how a spatially homogeneous and isotropic universe expands or contracts as a function of time. Friedmann published his first results, implying expanding or contracting space, five years before Lemaître interpreted the observed galaxy redshifts in terms of an expanding universe, and seven years before Hubble published Hubble's law.

Friedmann derived his eponymous equation starting from Einstein's field equation, using the full power of general relativity. Even without bringing relativity into play, some (though not all) of the aspects of the Friedmann equation can be understood with the use of purely Newtonian dynamics. To see how the expansion or contraction of the universe can be viewed from a Newtonian viewpoint, I will first derive the nonrelativistic equivalent of the Friedmann equation, starting from Newton's law of gravity and second law of motion. Then I will state (without proof) the modifications that must be made to find the more correct, general relativistic form of the Friedmann equation.

To begin, consider a homogeneous sphere of matter, with total mass M_s constant with time (Figure 4.1). The sphere is expanding or contracting isotropically, so that its radius $R_s(t)$ is increasing or decreasing with time. Place a test mass, of infinitesimal mass m, at the surface of the sphere. The gravitational force F experienced by the test mass will be, from Newton's law of gravity,

$$\boxed{F = -\frac{GM_s m}{R_s(t)^2} = m.a} \quad \boxed{a = -\frac{GM}{R_s(t)^2}} \tag{4.10}$$

[2] Using the Library of Congress transliteration system for Cyrillic, his name would be "Aleksandr Fridman." However, in the German scientific journals where he published his main results, he alternated between the spellings "Friedman" and "Friedmann" for his last name. The two-n spelling is more popular among historians of science.



$R_s(t) = a(t) \cdot r_s$
↑ Exp Factor (Time-dep) | ↑ Comoving Radial Coord (Space coord)

Figure 4.1 A sphere of radius $R_s(t)$ and mass M_s, expanding or contracting under its own gravity.

The gravitational acceleration at the surface of the sphere will then be, from Newton's second law of motion,

$$a = \frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}. \tag{4.11}$$

Multiply each side of the equation by dR_s/dt and integrate to find

H/W 4.1: Prove 4.12

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U, \tag{4.12}$$

where U is a constant of integration. Equation 4.12 simply states that the sum of the kinetic energy per unit mass,

$$\frac{d}{dt}\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \left(\frac{dR_s}{dt}\right)\left(\frac{d^2R_s}{dt^2}\right) = \frac{d}{dt}\frac{-GM_s}{R_s(t)} = \frac{-GM_s}{R_s^2}\cdot\frac{dR_s}{dt}$$

$$\boxed{\epsilon_{\rm kin} = \frac{1}{2}\left(\frac{dR_s}{dt}\right)^2, = \frac{E_{kin}}{m} = \frac{\frac{1}{2}mv^2}{m}} \tag{4.13}$$

and the gravitational *potential* energy per unit mass,

$$\boxed{\epsilon_{\rm pot} = -\frac{GM_s}{R_s(t)}, = \frac{E_{POT}}{m} = \frac{-GM_s \cdot m}{R_s(t) \cdot m}} \tag{4.14}$$

is constant for a bit of matter at the surface of a sphere, as the sphere expands or contracts under its own gravitational influence.

Since the mass of the sphere is constant, we may write

$$M_s = \frac{4\pi}{3}\rho(t)R_s(t)^3. \tag{4.15}$$

Since the expansion or contraction is isotropic about the sphere's center, we may write the radius $R_s(t)$ in the form

$$R_s(t) = \underbrace{a(t)}_{\text{Time-dep}}\,\underbrace{r_s}_{\text{Time-Independent!}}, \tag{4.16}$$

where $a(t)$ is the scale factor and r_s is the comoving radius of the sphere. In terms of $\rho(t)$ and $a(t)$, the energy conservation Equation 4.12 can be rewritten in the form

(4.12) + (4.16) } ⇒

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho(t)a(t)^2 + U. \tag{4.17}$$

Dividing each side of Equation 4.17 by $r_s^2 a^2/2$ yields the equation

H/W 4.2 PROVE 4.17+18, SOLVE (4.18) FOR 3 MEANINGFUL VALUES OF U

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}. \tag{4.18}$$

NEWTONIAN FORM OF FRIEDMANN EQUATION

Equation 4.18 gives the Friedmann equation in its Newtonian form.

Note that the time derivative of the scale factor only enters into Equation 4.18 as $\dot{a}^2$; a contracting sphere ($\dot{a} < 0$) is simply the time reversal of an expanding sphere ($\dot{a} > 0$). Let's concentrate on the case of an expanding sphere, analogous to the expanding universe in which we find ourselves. The future of the expanding sphere falls into one of three classes, depending on the sign of U. First, consider the case $U > 0$. In this case, the right-hand side of Equation 4.18 is always positive. Therefore, $\dot{a}^2$ is always positive, and the expansion of the sphere never stops. Second, consider the case $U < 0$. In this case, the right-hand side of Equation 4.18 starts out positive. However, at a maximum scale factor

EXTRA CREDIT: SHOW

$$a_{\max} = -\frac{GM_s}{Ur_s}, \tag{4.19}$$

only for $U<0$!

the right-hand side will equal zero, and expansion will stop. Since $\ddot{a}$ will still be negative, the sphere will then contract. Third, and finally, consider the case $U = 0$. This is the boundary case in which $\dot{a} \to 0$ as $t \to \infty$ and $\rho \to 0$.

The three possible fates of an expanding sphere in a Newtonian universe are analogous to the three possible fates of a ball thrown upward from the surface of the Earth. First, the ball can be thrown upward with a speed greater than the escape speed; in this case, the ball continues to go upward forever. Second, the ball can be thrown upward with a speed less than the escape speed; in this case, the ball reaches a maximum altitude, then falls back down. Third, and finally, the ball can be thrown upward with a speed exactly equal to the escape speed; in this case, the speed of the ball approaches zero as $t \to \infty$. $v > v_{esc}$

The Friedmann equation in its Newtonian form (Equation 4.18) is useful in picturing how isotropically expanding objects behave under the influence of their self-gravity. However, its application to the real universe must be regarded with considerable skepticism. First of all, a spherical volume of finite radius R_s cannot represent a homogeneous, isotropic universe. In a finite spherical volume, there exists a special location (the center of the sphere), violating the assumption of homogeneity, and at any point there exists a special

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direction (the direction pointing toward the center), violating the assumption of isotropy. What if we instead regard the sphere of radius R_s as being carved out of an infinite, homogeneous, isotropic universe? In that case, Newtonian dynamics tell us that the gravitational acceleration inside a hollow spherically symmetric shell is equal to zero. We divide up the region outside the sphere into concentric shells, and thus conclude that the test mass m at R_s experiences no net acceleration from matter at $R > R_s$. Unfortunately, a Newtonian argument of this sort assumes that space is perfectly Euclidean, an assumption that we can't necessarily make in the real universe. A derivation of the correct Friedmann equation has to begin, as Friedmann himself began, with Einstein's field equation.

$\frac{2U}{r_s^2 a(t)^2} = \frac{-kc^2}{R_0^2 a(t)^2}$

$= \frac{2U}{R_s^2}$

$\boxed{U = -\frac{1}{2}kc^2}$

The correct form of the Friedmann equation, including all general relativistic effects, is

(4.18) + $\varepsilon(t) = \rho(t)c^2$ $\Longrightarrow$

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}. \quad (4.20)$$

(4.18) + $U = -\frac{1}{2}kc^2$

RELATIVISTIC FORM OF FRIEDMANN EQ

$E = mc^2$

$\frac{E}{V} = \frac{mc^2}{V}$

Note the changes made in going from the Newtonian form of the Friedmann equation (Equation 4.18) to the correct relativistic form (Equation 4.20). The first change is that the mass density ρ has been replaced by an energy density ε divided by the square of the speed of light. One of Einstein's insights was that in determining the gravitational influence of a particle, the important quantity was not its mass m but its energy,

REST ENERGY — RELATIVISTIC MOMENTUM — $E = mc^2\left(1+\frac{v^2}{c^2}\right)^{1/2} \simeq$

$$E = (m^2c^4 + p^2c^2)^{1/2}. \quad (4.21)$$

$\Rightarrow \simeq mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \simeq mc^2 + \frac{1}{2}mv^2 +$

"NEWTON" $p = mv$

REST ENERGY — KIN. ENERGY

Here p is the momentum of the particle, as measured by an observer at the particle's location who sees the universe expanding isotropically. Any motion that a particle has, in addition to the motion associated with the expansion or contraction of the universe, is called the particle's *peculiar* motion.[3] If a massive particle is nonrelativistic, with a peculiar velocity $v \ll c$, then its peculiar momentum will be $p \approx mv$ and its energy will be

CLASSICAL

$$E_{\text{nonrel}} \approx mc^2(1 + v^2/c^2)^{1/2} \approx mc^2 + \frac{1}{2}mv^2 = mc^2 + E_{kin} \quad (4.22)$$

Thus, if the universe contained only massive, slowly moving particles, then the energy density ε would be nearly equal to ρc^2, with only a small correction for the kinetic energy $mv^2/2$ of the particles. However, photons and other massless particles also have an energy,

$$E_{\text{rel}} = pc = hf \Rightarrow p = \frac{h\nu}{c} = \frac{h}{\lambda} \Rightarrow \quad (4.23)$$

$\lambda_B = \frac{h}{p} = \frac{hc}{E}$

QUANTUM de Broglie λ

[3] The adjective "peculiar" comes from the Latin word *peculium*, meaning "private property." The peculiar motion of a particle is thus the motion that belongs to the particle alone, and not to the global expansion or contraction of the universe.

which contributes to the energy density ε. Not only do photons respond to the curvature of spacetime, they also contribute to it.

NEWTON: $E=\frac{1}{2}mv^2$

$U=\frac{E}{V}=-\frac{1}{2}kc^2$

CLASSIC ENERGY DENSITY RELATIV.

WHERE $k \simeq -\frac{m}{V}$

K IS LIKE AN ENERGY DENSITY WITH THE SIGN FLIPPED.

The second change that must be made in going from the Newtonian form of the Friedmann equation to the correct relativistic form is making the substitution

$$\boxed{u=-\frac{1}{2}kc^2} \Longleftrightarrow \boxed{\frac{2U}{r_s^2}=-\frac{\kappa c^2}{R_0^2}.} \tag{4.24}$$

$K=+1 \Rightarrow U<0$ POSITIVE CURVATURE
$K=0 \Rightarrow U=0$ FLAT
$K=-1 \Rightarrow U>0$ NEGATIVE

The Newtonian case with $U<0$ corresponds to the relativistic case with positive curvature ($\kappa=+1$); conversely, $U>0$ corresponds to negative curvature ($\kappa=-1$). The Newtonian special case with $U=0$ corresponds to the relativistic special case where the space is perfectly flat ($\kappa=0$). Although I have not given the derivation of the Friedmann equation in the general relativistic case, it makes sense that the curvature, given by κ and R_0, the expansion rate, given by $a(t)$, and the energy density ε should be bound up together in the same equation. After all, in Einstein's view, the energy density of the universe determines both the curvature of space and the overall dynamics of the expansion.

The Friedmann equation is a Very Important Equation in cosmology.[4] However, if we want to apply the Friedmann equation to the real universe, we must have some way of tying it to observable properties. For instance, the Friedmann equation can be tied to the Hubble constant, H_0. Remember, in a universe whose expansion (or contraction) is described by a scale factor $a(t)$, there's a linear relation between recession speed v and proper distance d:

$$v(t)=H(t)d(t), \tag{4.25}$$

where $H(t)\equiv\dot{a}/a$. Thus, the Friedmann equation can be rewritten in the form

$(4.20) \Rightarrow$
$$H(t)^2=\frac{8\pi G}{3c^2}\varepsilon(t)-\frac{\kappa c^2}{R_0^2a(t)^2}. \tag{4.26}$$

At the present moment,

$$H_0=H(t_0)=\left(\frac{\dot{a}}{a}\right)_{t=t_0}=68\pm2\,\mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{4.27}$$

(As an aside, the time-varying function $H(t)$ is generally known as the "Hubble parameter," while H_0, the value of $H(t)$ at the present day, is known as the "Hubble constant.") The Friedmann equation evaluated at the present moment is

HW 4.3: *SHOW (4.28a) + (4.28b)
*What special meaning for $k=0$?

$$H_0^2=\frac{8\pi G}{3c^2}\varepsilon_0-\frac{\kappa c^2}{R_0^2}, \tag{4.28 a}$$

using the convention that a subscript "0" indicates the value of a time-varying quantity evaluated at the present. Thus, the Friedmann equation gives a relation

*What is the critical density ρ_0 (cgs)? $H_0^2=\frac{8\pi G}{3c^2}\varepsilon_0=\frac{8\pi G}{3}\rho_0 \Rightarrow \rho_0=\frac{3H_0^2}{8\pi G}$ (4.28)b

4 You should consider writing it in reverse on your forehead so that you can see it every morning in the mirror when you comb your hair.

$\varepsilon_0=\rho_0c^2=3H_0^2c^2$

between the Hubble parameter H_0, which tells us the current rate of expansion, ε_0, which tells us the current energy density, and κ/R_0^2, which tells us the current curvature. (4.28) WITH $\varepsilon_0 = 0 \Rightarrow \kappa = -1 \Rightarrow R_0 = c/H_0$! ← NEGATIVE CURVATURE

In a spatially flat universe ($\kappa = 0$), the Friedmann equation takes a particularly simple form:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t). \quad \text{FLAT UNIVERSE} \tag{4.29}$$

Thus, for a given value of the Hubble parameter, there is a *critical density*,

$\varepsilon(t) > \varepsilon_c(t) \Rightarrow \kappa = +1$ CLOSED SPACE
$\varepsilon(t) = \varepsilon_c(t) \Rightarrow \kappa = 0$ FLAT "
$\varepsilon(t) < \varepsilon_c(t) \Rightarrow \kappa = -1$ OPEN "

$$\varepsilon_c(t) \equiv \frac{3c^2}{8\pi G}H(t)^2 \Rightarrow \varepsilon_c(t_0) = \frac{3c^2 H_0^2}{8\pi G} \tag{4.30}$$

If the energy density $\varepsilon(t)$ is greater than this value, the universe is positively curved ($\kappa = +1$). If $\varepsilon(t)$ is less than this value, the universe is negatively curved ($\kappa = -1$). Knowing the Hubble constant to within about 3%, we can compute the current value of the critical density to within about 6%:

$$\varepsilon_{c,0} = \frac{3c^2}{8\pi G}H_0^2 = (7.8 \pm 0.5) \times 10^{-10}\,\mathrm{J\,m^{-3}} = 4870 \pm 290\,\mathrm{MeV\,m^{-3}}. \tag{4.31}$$

The critical density is frequently written as the equivalent mass density,

$$\begin{aligned}\rho_{c,0} \equiv \frac{\varepsilon_{c,0}}{c^2} &= (8.7 \pm 0.5) \times 10^{-27}\,\mathrm{kg\,m^{-3}} \\ &= (1.28 \pm 0.08) \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}}.\end{aligned} \tag{4.32}$$

The current critical density is roughly equivalent to a density of one proton per 200 liters. This is definitely not a large density, by terrestrial standards; a 200-liter drum filled with liquid water, for instance, contains $\sim 10^{29}$ protons and neutrons. The critical density is not even a large density by the standards of interstellar space within our galaxy; even the hottest, most tenuous regions of the interstellar medium have a few protons per liter. However, keep in mind that most of the volume of the universe consists of intergalactic voids, where the density is extraordinarily low. When averaged over scales of 100 Mpc or more, the mean energy density of the universe, as it turns out, is very close to the critical density.

When discussing the curvature of the universe, it is useful to talk about the energy density in terms of the dimensionless *density parameter*

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)} = \frac{8\pi G}{3c^2 H(t)^2}\cdot\varepsilon(t) \tag{4.33}$$

The current value of the density parameter, determined from a combination of observational data, lies in the range $0.995 < \Omega_0 < 1.005$. Written in terms of the density parameter Ω, the Friedmann equation becomes

$\left.\begin{array}{l}(4.20)/H(t)^2 \\ +(4.33)\end{array}\right\} \rightarrow$

$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}. \tag{4.34}$$

Note that, since the right-hand side of Equation 4.34 cannot change sign as the universe expands, neither can the left-hand side. If $\Omega < 1$ at any time, it remains less than one for all time; similarly, if $\Omega > 1$ at any time, it remains greater than one for all time, and if $\Omega = 1$ at any time, $\Omega = 1$ at all times. At the present moment, the relation between curvature, density, and expansion rate can be written in the form

HW 4.4: * SHOW (4.35)+(4.36)

$$1 - \Omega_0 = -\frac{\kappa c^2}{R_0^2 H_0^2}, \Rightarrow \begin{cases} \text{ONCE } \Omega > 1 \Rightarrow \text{ALWAYS } \Omega > 1 \ (k=+1) \\ \text{ONCE } \Omega \equiv 1 \Rightarrow \text{ALWAYS } \Omega = 1 \ (k=0) \\ \text{ONCE } \Omega < 1 \Rightarrow \text{ALWAYS } \Omega < 1 \ (k=-1) \end{cases} \tag{4.35}$$

* DISCUSS ITS CRITICAL IMPLICATIONS)

or

(LINK TO INFLATION)

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1). \tag{4.36}$$

If you know Ω_0, you know the sign of the curvature (κ). If, in addition, you know the Hubble distance, c/H_0, you can compute the radius of curvature (R_0).

HINT: Use (4.26)+(4.33) to express $\Omega(t)$ as function of $(1+z)$, and so, prove that: $\boxed{\left(1 - \frac{1}{\Omega(t)}\right) = (1+z)^{-1}\left(1 - \frac{1}{\Omega_0}\right)}$ or $\boxed{\frac{\Omega(t)-1}{\Omega(t)} = \left(\frac{1}{1+z}\right)\left(\frac{\Omega_0 - 1}{\Omega_0}\right)}$

MAY DERIVE THIS FOR EXTRA CREDIT, BUT NEED TO DISCUSS ITS GREAT IMPORTANCE IN ANY CASE

4.3 The Fluid and Acceleration Equations

Although the Friedmann equation is indeed important, it cannot, all by itself, tell us how the scale factor $a(t)$ evolves with time. Even if we had accurate boundary conditions (precise values for ε_0 and H_0, for instance), it still remains a single equation in two unknowns, $a(t)$ and $\varepsilon(t)$.

We need another equation involving a and ε if we are to solve for a and ε as functions of time. The Friedmann equation, in the Newtonian approximation, is a statement of energy conservation; in particular, it says that the sum of the gravitational potential energy and the kinetic energy of expansion is constant. Energy conservation is a generally useful concept in Newtonian physics, so let's look at another manifestation of the same concept – the first law of thermodynamics:

Isotropy + Homogeneity ⟺
⟺ No Bulk heat flow ⟺ $dQ=0$

$$\boxed{dQ = dE + P dV,} \tag{4.37}$$

⟺ ADIABATIC
⟺ ENTROPY

where dQ is the heat flow into or out of a volume, dE is the change in internal energy, P is the pressure, and dV is the change in volume. This equation was applied in Section 2.5 to a comoving volume filled with photons, but it applies equally well to a comoving volume filled with any sort of fluid. If the universe is perfectly homogeneous, then for any volume $dQ = 0$; that is, there is no bulk flow of heat.

$dS = \frac{dQ}{T} = 0$

Processes for which $dQ = 0$ are known as *adiabatic* processes. The term "adiabatic" comes from the Greek word *adiabatos*, meaning "not to be passed through," referring to the fact that heat does not pass through the boundary of the volume.[5] Saying that the expansion of the universe is adiabatic is also a statement about entropy. The change in entropy dS within a region is given by the relation $dS = dQ/T$; thus, an adiabatic process is one in which entropy is not increased. A homogeneous, isotropic expansion of the universe does not increase the universe's entropy.

Since $dQ = 0$ for a comoving volume as the universe expands, the first law of thermodynamics, as applied to the expanding universe, reduces to the form

$\frac{d}{dt}(4.37) \Rightarrow \frac{dQ}{dt} = \frac{dE}{dt} + P\frac{dV}{dt} \Longrightarrow$

$$\dot{E} + P\dot{V} = 0. \tag{4.38}$$

For concreteness, consider a sphere of comoving radius r_s expanding along with the universal expansion, so that its proper radius is $R_s(t) = a(t)r_s$. The volume of the sphere is

$$V(t) = \frac{4\pi}{3} r_s^3 a(t)^3, \tag{4.39}$$

so the rate of change of the sphere's volume is

$$\dot{V} = \frac{4\pi}{3} r_s^3 (3a^2\dot{a}) = V\left(3\frac{\dot{a}}{a}\right) \boxed{= 3V \cdot H(t) = \dot{V}} \tag{4.40}$$

Volume factor

The internal energy of the sphere is

$$E(t) = V(t)\varepsilon(t), \tag{4.41}$$

so the rate of change of the sphere's internal energy is

$$\dot{E} = V\dot{\varepsilon} + \dot{V}\varepsilon = V\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}\varepsilon\right). \tag{4.42}$$

Combining Equations 4.38, 4.40, and 4.42, we find that the first law of thermodynamics in an expanding (or contracting) universe takes the form

$$V\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}\varepsilon + 3\frac{\dot{a}}{a}P\right) = 0, \tag{4.43}$$

Energy density term

Pressure term

or

FLUID EQUATION (OR CONTINUITY EQUATION)

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0. \Rightarrow \boxed{\dot{\varepsilon}\frac{a}{\dot{a}} = -3(\varepsilon + P)} \tag{4.44}$$

$= \frac{\dot{\varepsilon}}{H(t)}$

This equation is called the *fluid equation*, and is the second of the key equations describing the expansion of the universe.[6] Unlike the Friedmann equation, whose relativistic form is different from its Newtonian form, the fluid equation is unchanged by the switch from Newtonian physics to general relativity.

[5] The word "adiabatic" is thus etymologically related to "diabetes," a word that refers to the quick passage of liquids through the human body, creating the increased thirst and frequent urination that are symptomatic of untreated diabetes. If you do not have diabetes, you could, I suppose, refer to yourself as "a-diabetic."

[6] Write it on your forehead just underneath the Friedmann equation.

The Friedmann equation and fluid equation can be combined into an acceleration equation that tells how the universe speeds up or slows down with time. The Friedmann equation (Equation 4.20), multiplied by a^2, takes the form

4.20

$$H(t)^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2} \Rightarrow \quad \dot a^2 = \frac{8\pi G}{3c^2}\varepsilon a^2 - \frac{\kappa c^2}{R_0^2}. \tag{4.45}$$

CONSTANT!

Taking the time derivative yields

$$2\dot a\ddot a = \frac{8\pi G}{3c^2}(\dot\varepsilon a^2 + 2\varepsilon a\dot a). \tag{4.46}$$

Dividing by $2\dot a a$ tells us

$$\frac{\ddot a}{a} = \frac{4\pi G}{3c^2}\left(\dot\varepsilon\frac{a}{\dot a} + 2\varepsilon\right). \tag{4.47}$$

Using the fluid equation (Equation 4.44), we may make the substitution

$$\dot\varepsilon\frac{a}{\dot a} = -3(\varepsilon + P) \tag{4.48}$$

to find the usual form of the *acceleration equation*,

ACCELLERATION EQUATION:

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P).} \tag{4.49}$$

SEMI CLASSIC ρ (= Classical analog of the energy-momentum tensor in GR!

If the energy density ε is positive, then it provides a negative acceleration – that is, it decreases the value of $\dot a$ and reduces the relative velocity of any two points in the universe. The acceleration equation also includes the pressure P associated with the material filling the universe.[7]

A gas made of ordinary baryonic matter has a positive pressure P, resulting from the random thermal motions of the molecules, atoms, or ions of which the gas is made. A gas of photons also has a positive pressure, as does a gas of neutrinos or WIMPs. The positive pressure associated with these components of the universe will cause the expansion to slow down. Suppose, though, that the universe had a component with $\varepsilon > 0$ and

$$\boxed{P < -\frac{1}{3}\varepsilon.} \tag{4.50}$$

$\Rightarrow P$ WINS OVER ε in (4.49)!

Inspection of the acceleration equation (Equation 4.49) shows us that such a component will cause the expansion of the universe to speed up rather than slow down.

(For $\Lambda \neq 0$, we will get $P = -\varepsilon$)

4.4 Equations of State

To recap, we now have three key equations that describe how the universe expands. There's the Friedmann equation,

~ MOST IMPORTANT THOUGHT EINSTEIN

[7] Although we think of ε as an energy per unit volume and P as a force per unit area, they both have the same dimensionality: $1\,\mathrm{J\,m^{-3}} = 1\,\mathrm{N\,m^{-2}} = 1\,\mathrm{kg\,m^{-1}\,s^{-2}}$.

FRIEDMANN EQ (4.20):
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}, = H(t)^2 \qquad (4.51)$$

the fluid equation,

FLUID EQUATION (4.44): CONTINUITY EQ.
$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0, \qquad (4.52)$$

and the acceleration equation,

ACCELERATION EQ (4.49):
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P). \qquad (4.53)$$

Of these three equations, only two are independent, because Equation 4.53, as we've just seen, can be derived from Equations 4.51 and 4.52. Thus, we have a system of two independent equations in three unknowns – the functions $a(t)$, $\varepsilon(t)$, and $P(t)$. To solve for the scale factor, energy density, and pressure as a function of cosmic time, we need another equation. What we need is an *equation of state*; that is, a mathematical relation between the pressure and energy density of the stuff that fills up the universe. If only we had a relation of the form

$$\boxed{P = P(\varepsilon),} \qquad (4.54)$$

life would be complete – or at least, our set of equations would be complete. We could then, given the appropriate boundary conditions, solve them to find how the universe expanded in the past, and how it will expand (or contract) in the future.

In general, equations of state can be dauntingly complicated. Condensed matter physicists frequently deal with substances in which the pressure is a complicated nonlinear function of the density. Fortunately, cosmology usually deals with dilute gases, for which the equation of state is simple. For substances of cosmological importance, the equation of state can be written in a simple linear form:

$$\boxed{P = w\varepsilon,} \Rightarrow w \equiv \frac{P}{\varepsilon} = \frac{P}{\rho c^2} = \frac{kT}{\mu c^2} \qquad (4.55) = w$$

where w is a dimensionless number.

Consider, for instance, a low-density gas of nonrelativistic massive particles. Nonrelativistic, in this case, means that the random thermal motions of the gas particles have peculiar velocities which are tiny compared to the speed of light. Such a nonrelativistic gas obeys the perfect gas law,

IDEAL GAS:
$$PV = NkT \Rightarrow P = \frac{N\mu}{V/\mu}kT \Longrightarrow \quad P = \frac{\rho}{\mu}kT = \frac{kT}{\mu c^2}.\varepsilon = P = w.\varepsilon \qquad (4.56)$$

where μ is the mean mass of the gas particles. The energy density ε of a nonrelativistic gas is almost entirely contributed by the mass of the gas particles: $\varepsilon \approx \rho c^2$. Thus, in terms of ε, the perfect gas law is

$$\rho = \frac{M}{V} = \frac{N\mu}{V} \qquad \rho = \frac{\varepsilon}{c^2} \text{ since } \varepsilon = \frac{mc^2}{V} = \rho c^2$$

Using (4.54): $P = w\varepsilon \Longrightarrow$

$$P \approx \frac{kT}{\mu c^2}\varepsilon = w \cdot \varepsilon \Leftrightarrow w = \frac{kT}{\mu c^2} \quad (4.57)$$

For a nonrelativistic gas, the temperature T and the root mean square thermal velocity $\langle v^2 \rangle$ are associated by the relation

$$3kT = \mu \langle v^2 \rangle . \Leftarrow \frac{3}{2}kT = \frac{1}{2}\mu \langle v^2 \rangle \quad (4.58)$$

HW 4.5 DISCUSS

* Meaning of $w < 1$
* $w \simeq 0 \rightarrow$ NON-REL
* $w = \frac{1}{3} \rightarrow$ REL (PHOTON GAS)
* $w < -\frac{1}{3} \rightarrow$ ACCELERATING: $-\frac{1}{3}$ for cosmic strings, $-\frac{2}{3}$ for domain walls
* $w = -1 \Rightarrow \Lambda > 0$

Thus, the equation of state for a nonrelativistic gas can be written in the form

$$P_{\text{nonrel}} = w\varepsilon_{\text{nonrel}} = \frac{kT}{\mu c^2} \cdot \varepsilon_{nr} = \frac{\mu \langle v^2 \rangle / 3}{\mu c^2} \cdot \varepsilon_{nr} \quad (4.59)$$

$$\Downarrow$$

where

$$w \approx \frac{\langle v^2 \rangle}{3c^2} \ll 1. \qquad w_{nr} \simeq \frac{\langle v^2 \rangle}{3c^2} \quad (4.60)$$

Most of the gases we encounter in everyday life are nonrelativistic. For instance, in air at room temperature, nitrogen molecules are slow-poking along with a root mean square velocity of $\sim 500\,\text{m s}^{-1}$, yielding $w \sim 10^{-12}$. Even in astronomical contexts, gases are mainly nonrelativistic at the present moment. Within a gas of ionized hydrogen, for instance, the electrons are nonrelativistic as long as $T \ll 6 \times 10^9$ K; the protons are nonrelativistic when $T \ll 10^{13}$ K.

A gas of photons, or other massless particles, is guaranteed to be relativistic. Although photons have no mass, they have momentum, and hence exert pressure. The equation of state of photons, or of any other relativistic gas, is

(4.60) + $v = c$ $\Longrightarrow$

$$P_{\text{rel}} = \frac{1}{3}\varepsilon_{\text{rel}}. \quad (4.61)$$

(This relation has already been used in Section 2.5, to compute how the cosmic microwave background cools as the universe expands.) A gas of highly relativistic massive particles (with $\langle v^2 \rangle \sim c^2$) will also have $w = 1/3$; a gas of mildly relativistic particles (with $0 < \langle v^2 \rangle < c^2$) will have $0 < w < 1/3$.

Some values of w are of particular interest. For instance, the case $w = 0$ is of interest, because we know that our universe contains nonrelativistic matter. The case $w = 1/3$ is of interest, because we know that our universe contains photons. For simplicity, we will refer to the component of the universe that consists of nonrelativistic particles (and hence has $w \approx 0$) as "matter," and the component that consists of photons and other relativistic particles (and hence has $w = 1/3$) as "radiation." The case $w < -1/3$ is of interest, because a component with $w < -1/3$ provides a positive acceleration ($\ddot{a} > 0$ in Equation 4.53). A component of the universe with $w < -1/3$ is referred to generically as "dark energy" (a phrase coined by the cosmologist Michael Turner). One form of dark energy is of special interest; observational evidence,

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which we'll review in future chapters, indicates that our universe may contain a *cosmological constant.* A cosmological constant may be defined simply as a component of the universe that has $w = -1$, and hence has $P = -\varepsilon$. The cosmological constant, also designated by the Greek letter Λ, has had a controversial history. To learn why cosmologists have had such a long-standing love/hate affair with the cosmological constant Λ, it is necessary to make a brief historical review.

Λ has $w \equiv -1$

4.5 Learning to Love Lambda

The cosmological constant Λ was first introduced by Albert Einstein. After publishing his first paper on general relativity in 1915, Einstein, naturally enough, wanted to apply his field equation to the real universe. He looked around, and noted that the universe contains both radiation and matter. Since Einstein, along with every other earthling of his time, was unaware of the existence of the cosmic microwave background, he thought that most of the radiation in the universe was in the form of starlight. He also noted, quite correctly, that the energy density of starlight in our galaxy is much less than the rest energy density of the stars. Thus, Einstein concluded that the primary contribution to the energy density of the universe was from nonrelativistic matter, and that he could safely make the approximation that we live in a pressureless universe.

So far, Einstein was on the right track. However, in 1915, astronomers were unaware of the existence of the expansion of the universe. In fact, it was by no means settled that galaxies besides our own actually existed. After all, the sky is full of faint fuzzy patches of light. It took some time to sort out that some of the faint fuzzy patches are glowing clouds of gas within our galaxy and that some of them are galaxies in their own right, far beyond our own galaxy. Thus, when Einstein asked, "Is the universe expanding or contracting?" he looked, not at the motions of galaxies, but at the motions of stars within our galaxy. Einstein noted that some stars are moving toward us and that others are moving away from us, with no evidence that the galaxy is expanding or contracting.

The incomplete evidence available to Einstein led him to the belief that the universe is static – neither expanding nor contracting – and that it has a positive energy density but negligible pressure. Einstein then had to ask the question, "Can a universe filled with nonrelativistic matter, and nothing else, be static?" The answer to this question is "No!" A universe containing nothing but matter must, in general, be either expanding or contracting. The reason why this is true can be illustrated in a Newtonian context. If the mass density of the universe is ρ, then the gravitational potential Φ is given by Poisson's equation:

$\frac{\partial^2}{\partial r^2}\left(\frac{GM}{r}\right)$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{4.62}$$

The gravitational acceleration $\vec{a}$ at any point in space is then found by taking the gradient of the potential:

HW 4.6 : PROVE (4.62), (4.63), (4.64) and that $\Lambda = 4\pi G\rho$, discuss

DON'T DO (OR XTRA CREDIT)

$$\vec{a} = -\vec{\nabla}\Phi. \tag{4.63}$$

In a permanently static universe, $\vec{a}$ must vanish everywhere, implying the potential Φ must be constant in space. However, if Φ is constant, then (from Equation 4.62)

$$\rho = \frac{1}{4\pi G}\nabla^2\Phi = 0. \tag{4.64}$$

The only permissible static universe, in this analysis, is a totally empty universe. If you create a matter-filled universe that is initially static, then gravity will cause it to contract. If you create a matter-filled universe that is initially expanding, then it will either expand forever (if the Newtonian energy U is greater than or equal to zero) or reach a maximum radius and then collapse (if $U < 0$). Trying to make a matter-filled universe that doesn't expand or collapse is like throwing a ball into the air and expecting it to hover there.

How did Einstein surmount this problem? How did he reconcile the fact that the universe contains matter with his desire for a static universe? Basically, he added a fudge factor to the equations. In Newtonian terms, what he did was analogous to rewriting Poisson's equation in the form

$$\nabla^2\Phi + \Lambda = 4\pi G\rho. \tag{4.65}$$

The new term, symbolized by the Greek letter Λ, came to be known as the *cosmological constant*. Introducing Λ into Poisson's equation allows the universe to be static if you set $\Lambda = 4\pi G\rho$. $\Longleftrightarrow \nabla^2\Phi = 0$ IF AND ONLY IF

In general relativistic terms, what Einstein did was to add an additional term, involving Λ, to his field equation. If the Friedmann equation is re-derived from Einstein's field equation, with the Λ term added, it becomes

GENERALIZED FRIEDMANN EQUATION (4.20) WITH $\Lambda \neq 0$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} = H(t)^2 \tag{4.66}$$

The fluid equation is unaffected by the presence of a Λ term, so it still has the form

4.44 =

FLUID EQUATION (4.52) $\Rightarrow$ (OR CONTINUITY EQUATION)

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0. \tag{4.67}$$

With the Λ term present, the acceleration equation becomes

GENERALIZED ACCELLERATION EQUATION (4.53):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda}{3}. \tag{4.68}$$

A look at the Friedmann equation (Equation 4.66) tells us that adding the Λ term is equivalent to adding a new component to the universe with energy density

$$\varepsilon_\Lambda \equiv \frac{c^2}{8\pi G}\Lambda. \Longrightarrow \Lambda = \frac{8\pi G}{c^2}\varepsilon_\Lambda \tag{4.69}$$

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If Λ remains constant with time, then so does its associated energy density ε_Λ. The fluid equation (Equation 4.67) tells us that to have ε_Λ constant with time, the Λ term must have an associated pressure $\Rightarrow \dot{\varepsilon}_\Lambda \equiv 0$

(4.67) WITH $\varepsilon_\Lambda = \text{CONST} \Rightarrow \dot{\varepsilon}_\Lambda = 0$ $\Rightarrow w_\Lambda \varepsilon_\Lambda$ WITH $w_\Lambda = -1$

$$P_\Lambda = -\varepsilon_\Lambda = -\frac{c^2}{8\pi G}\Lambda. \iff w_\Lambda = \frac{P_\Lambda}{\varepsilon_\Lambda} \equiv -1 \qquad (4.70)$$

Thus, we can think of the cosmological constant as a component of the universe, which has a constant density ε_Λ and a constant pressure $P_\Lambda = -\varepsilon_\Lambda$. NEGATIVE PRESSURE!

By introducing a Λ term into his equations, Einstein got the static model universe he wanted. For the universe to remain static, both $\dot{a}$ and $\ddot{a}$ must be equal to zero. If $\ddot{a} = 0$, then in a universe with matter density ρ and cosmological constant Λ, the acceleration equation (Equation 4.68) reduces to

4.68 WITH $\ddot{a} = 0$, $\varepsilon = \rho c^2$ $\Rightarrow$

$$0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}. \quad (P=0) \qquad (4.71)$$

Thus, Einstein had to set $\Lambda = 4\pi G\rho$ in order to produce a static universe, just as in the Newtonian case. If $\dot{a} = 0$, the Friedmann equation (Equation 4.66) reduces to

STATIC $\dot{a} = 0$ $\Rightarrow a = \text{CONST} (=1)$

$$0 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{R_0^2} + \frac{\Lambda}{3} = 4\pi G\rho - \frac{\kappa c^2}{R_0^2} \Rightarrow \frac{4\pi G\rho}{c^2} = \frac{\kappa}{R_0^2} \Rightarrow R_0^2 = \frac{\kappa c^2}{4\pi G\rho} \qquad (4.72)$$

$R_0 = \frac{c\sqrt{\kappa}}{\sqrt{4\pi G\rho}} = c\sqrt{\frac{\kappa}{\Lambda}}$

Einstein's static model therefore had to be positively curved ($\kappa = +1$), with a radius of curvature

$\Lambda \simeq 0.68$ FOR $\kappa = +1 \Rightarrow R_0 = \frac{c}{\sqrt{\Lambda}}$

R_0.

$$R_0 = \frac{c}{2(\pi G\rho)^{1/2}} = \frac{c}{\Lambda^{1/2}}. = \frac{c}{\sqrt{0.68}} \simeq 1.21 \left(\frac{c}{H_0}\right) \qquad (4.73)$$

IN UNITS OF c/H_0

Although Einstein published the details of his static, positively curved, matter-filled model in 1917, he was dissatisfied with the model. He believed that the cosmological constant was "gravely detrimental to the formal beauty of the theory." In addition to its aesthetic shortcomings, the model had a practical defect; it was unstable. Although Einstein's static model was in equilibrium, with the repulsive force of Λ balancing the attractive force of ρ, it was an unstable equilibrium. Consider expanding Einstein's universe just a tiny bit. The energy density of Λ remains unchanged, but the energy density of matter drops. Thus, the repulsive force is greater than the attractive force, and the universe expands further. This causes the matter density to drop further, which causes the expansion to accelerate, which causes the matter density to drop further, and so forth. Expanding Einstein's static universe triggers runaway expansion; similarly, compressing it causes a runaway collapse.

Einstein was willing, even eager, to dispose of the "ugly" cosmological constant in his equations. Hubble's 1929 paper on the redshift–distance relation gave Einstein the necessary excuse for tossing Λ onto the rubbish heap.[8]

[8] According to the physicist George Gamow, writing his memoirs much later, Einstein "remarked that the introduction of the cosmological term [Λ] was the biggest blunder of his life."

Ironically, however, the same paper that caused Einstein to abandon the cosmological constant caused other scientists to embrace it. In his initial analysis, Hubble underestimated the distance to galaxies, and hence overestimated the Hubble constant. Hubble's initial value of $H_0 = 500\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ leads to a Hubble time $H_0^{-1} = 2\,\mathrm{Gyr}$. However, by the year 1929, the technique of radiometric dating, as pioneered by the geologist Arthur Holmes, was indicating that the Earth was $\sim 3\,\mathrm{Gyr}$ old. How could cosmologists reconcile a short Hubble time with an old Earth? Some cosmologists pointed out that one way to increase the age of the universe for a given value of H_0^{-1} was to introduce a cosmological constant. If the value of Λ is large enough to make $\ddot{a} > 0$, then $\dot{a}$ was smaller in the past than it is now, and consequently the universe is older than H_0^{-1}.

If Λ has a value greater than $4\pi G\rho_0$, then the expansion of the universe is accelerating, and the universe can be arbitrarily old for a given value of H_0^{-1}. Since 1917, the cosmological constant has gone in and out of fashion, like sideburns or short skirts. It has been particularly fashionable during periods when the favored value of the Hubble time H_0^{-1} has been embarrassingly short compared to the estimated ages of astronomical objects. Currently, the cosmological constant is popular, thanks to observations (discussed in Section 6.5) indicating that the expansion of the universe has a positive acceleration.

A question that has been asked since the time of Einstein – and one which we've assiduously dodged until this moment – is "What is the physical cause of the cosmological constant?" In order to give Λ a physical meaning, we need to identify some component of the universe whose energy density ε_Λ remains constant as the universe expands or contracts. Currently, a leading candidate for this component is the *vacuum energy*.

In classical physics, the idea of a vacuum having energy is nonsense. A vacuum, from the classical viewpoint, contains nothing; and as King Lear would say, "Nothing can come of nothing." In quantum physics, however, a vacuum is not a sterile void. The Heisenberg uncertainty principle permits particle–antiparticle pairs to spontaneously appear and then annihilate in an otherwise empty vacuum. The total energy ΔE and the lifetime Δt of these pairs of virtual particles must satisfy the relation[9]

$\gtrsim$ ALSO WRITTEN AS: TYPO!

$\Delta E, \Delta t$ = FINITE $\Leftrightarrow$

E, t NOT BOTH EXACTLY KNOWN $\Leftrightarrow$

$$\boxed{\Delta E \Delta t \cancel{\leq} h.} \quad (4.74)$$

$\Leftrightarrow \Delta x \Delta p \gtrsim \hbar$

$\Leftrightarrow$ x, p NOT BOTH EXACTLY KNOWN

Just as there's an energy density associated with the real particles in the universe, there is an energy density $\varepsilon_{\mathrm{vac}}$ associated with the virtual particle–antiparticle pairs. The vacuum density $\varepsilon_{\mathrm{vac}}$ is a quantum phenomenon that doesn't give a hoot

[9] The usual analogy that's made is to an embezzling bank teller who takes money from the till but who always replaces it before the auditor comes around. Naturally, the more money a teller is entrusted with, the more frequently the auditor makes random checks.

about the expansion of the universe and is independent of time as the universe expands or contracts.

Unfortunately, computing the numerical value of ε_{vac} is an exercise in quantum field theory that has not yet been successfully completed. It has been suggested that the natural value for the vacuum energy density is the Planck energy density,

$$\varepsilon_{\text{vac}} \sim \frac{E_P}{\ell_P^3} = \frac{M_P c^2}{\ell_P^3} (???) \simeq \frac{\left(\frac{\hbar c}{G}\right)^{1/2} c^2}{\left(\frac{G\hbar}{c^3}\right)^{3/2}} \simeq \frac{c^7}{\hbar G^2} \leftarrow \text{(H/W 1.1)} \tag{4.75}$$

As we've seen in Chapter 1, the Planck energy is large by particle physics standards ($E_P = 1.22 \times 10^{28}\,\text{eV} = 540$ kilowatt-hours), while the Planck length is small by anybody's standards ($\ell_P = 1.62 \times 10^{-35}\,\text{m}$). This gives an energy density (4.69) $\Longrightarrow \varepsilon_\Lambda = \frac{c^2}{8\pi G}\Lambda = 3\times10^{9}\,\text{eV m}^{-3}$ for $\Lambda_0 = 0.68$ and $1\,\text{J} = 6.24\times10^{18}\,\text{eV}$

$$\varepsilon_{\text{vac}} \sim 3 \times 10^{132}\,\text{eV m}^{-3} \quad (!!!) \simeq 10^{123}\varepsilon_0 \tag{4.76}$$

This is 123 orders of magnitude larger than the current critical density for our universe, and represents a spectacularly bad match between theory and observations. Obviously, we don't know much yet about the energy density of the vacuum! This is a situation where astronomers can help particle physicists, by deducing the value of ε_Λ from observations of the expansion of the universe. By looking at the universe at extremely large scales, we are indirectly examining the structure of the vacuum on extremely small scales.

Exercises

4.1 Suppose the energy density of the cosmological constant is equal to the present critical density $\varepsilon_\Lambda = \varepsilon_{c,0} = 4870\,\text{MeV m}^{-3}$. What is the total energy of the cosmological constant within a sphere 1 AU in radius? What is the rest energy of the Sun ($E_\odot = M_\odot c^2$)? Comparing these two numbers, do you expect the cosmological constant to have a significant effect on the motion of planets within the solar system?

4.2 Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.

4.3 If $\rho = 2.7 \times 10^{-27}\,\text{kg m}^{-3}$, what is the radius of curvature R_0 of Einstein's static universe? How long would it take a photon to circumnavigate such a universe?

4.4 Suppose that the universe were full of regulation baseballs, each of mass $m_{\text{bb}} = 0.145\,\text{kg}$ and radius $r_{\text{bb}} = 0.0369\,\text{m}$. If the baseballs were distributed uniformly throughout the universe, what number density of baseballs would